

A Seifert-van Kampen Theorem for Legendrian Submanifolds and Exact Lagrangian Cobordisms

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Abstract

We prove a Seifert-van Kampen theorem for Legendrian submanifolds and exact Lagrangian cobordisms, and use it to calculate the change in the DGA caused by critical Legendrian ambient surgery.

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1 Introduction

Let Y be an $(2n + 1)$ -dimensional manifold, and let $\zeta \subset TY$ be a hyperplane distribution. We say the pair (Y, ζ) is a **contact manifold** if ζ is maximally non-integrable. The primary example of a contact manifold we will use is the **one-jet manifold of M** , $J^1(M) = (T^*M \times \mathbb{R}, \zeta)$, $\zeta = \ker \alpha$, where $\alpha = dz - \theta$, z is the coordinate of \mathbb{R} and θ is the pullback to $J^1(M)$ of the tautological one-form of T^*M . Let $n = \dim M$; we say an n -dimensional submanifold $\Lambda \subset J^1(M)$ is **Legendrian** if $T\Lambda \subset \zeta$. This is equivalent to:

$$\alpha|_{\Lambda} = 0$$

We define the **Lagrangian projection** $\pi_{\mathbb{C}}$, **front projection** π_F , and **base projection** π_M to be the projections:

$$\pi_{\mathbb{C}} : J^1(M) \rightarrow T^*M$$

$$\pi_F : J^1(M) \rightarrow M \times \mathbb{R}$$

$$\pi_M : J^1(M) \rightarrow M$$

Let X be a $(2n)$ -dimensional manifold, and let ω be a two-form on X . We say ω is a **symplectic form** and X is a **symplectic manifold** if ω is closed and non-degenerate. We say (X, ω) is **exact symplectic** if ω is exact. The **symplecticization** of $J^1(M)$ is given by $(J^1(M) \times \mathbb{R}, \omega = d\beta)$, $\beta = e^t \alpha$, where t is the coordinate of \mathbb{R} ; this is then a symplectic manifold. We say an $(n + 1)$ -dimensional submanifold $L \subset J^1(M) \times \mathbb{R}$ is **Lagrangian** if:

$$\omega|_L = 0$$

We say L is **exact Lagrangian** if:

$$\beta|_L = df$$

For some function f .

We say two Legendrian submanifolds are **Legendrian isotopic** if they are isotopic through Legendrians, and similarly we say two exact Lagrangian submanifolds are **exact Lagrangian isotopic** if they are isotopic through exact

Lagrangians. We are interested in calculating invariants of compact Legendrians, one of which is the **Legendrian Contact Homology**, which is the homology of a unital graded algebra $\mathcal{A}_K(\Lambda)$ over a field K generated by the Reeb chords of Λ and with a differential given by counting rigid pseudoholomorphic disks. (We will ordinarily assume $K = \mathbb{Z}_2$, in which case we will suppress the subscript, but our primary results hold for arbitrary field.) We first prove a Seifert-van Kampen theorem for Legendrian Contact Homology. If $U \subset M$, we define $\mathcal{A}(\Lambda)|_U$ to be $\mathcal{A}(\Lambda)$ restricted to Reeb chords lying over U . This may or may not be a well-defined differential graded algebra. However:

Theorem 1.1. *Let $\Lambda \subset J^1(M)$ be a compact Legendrian submanifold, and let $S \subset M$ be a hypersurface that divides M into two components, R_1 and R_2 . Let N be an arbitrarily small neighborhood of S , and let $Q_i = R_i \cup N$. Then, after a Legendrian isotopy that does not change Λ outside of $\pi_M^{-1}(N)$, $\mathcal{A}_K(\Lambda)|_N, \mathcal{A}_K(\Lambda)|_{Q_i}$ are well-defined differential graded algebras, and the following diagram is a push-out square:*

$$\begin{array}{ccc} \mathcal{A}_K(\Lambda) & \xleftarrow{i_1} & \mathcal{A}_K(\Lambda)|_{Q_1} \\ i_2 \uparrow & & \uparrow j_1 \\ \mathcal{A}_K(\Lambda)|_{Q_2} & \xleftarrow{j_2} & \mathcal{A}_K(\Lambda)|_N \end{array}$$

Where i_1, i_2, j_1, j_2 are the inclusion maps.

The Legendrian isotopy in theorem 1.1 is called the **pinching isotopy** (also referred to as **dipping** in [13], and similar to the **splashing** of [8]). This result has been known for $\dim \Lambda = 1$ since [14], and for $\dim \Lambda = 2$ and for higher dimensions where the front projection has no codimension-2 singularities since [9]. We prove it for all cases.

Note that, since pinching does not change Λ outside of N , we can apply multiple pinching to separate Λ into more than two components. In addition, we can pinch inside the pinch zone, to separate N itself into multiple components.

This separation also descends to the linearized contact homologies if they exist, providing a Mayer-Veitoris theorem:

Theorem 1.2. *Let $\Lambda \subset J^1(M)$ be a compact Legendrian submanifold that has been pinched along a neighborhood N of a hyper surface S that divides M into Q_1, Q_2 . If Λ has an augmentation ϵ , then ϵ induces augmentations $\epsilon_N, \epsilon_{Q_i}$ of $\mathcal{A}(\Lambda)|_N, \mathcal{A}(\Lambda)|_{Q_i}$, and there is a long exact sequence:*

$$\begin{aligned} & \dots \rightarrow LCH_k(\mathcal{A}_K(\Lambda)|_N, \partial, \epsilon_N) \rightarrow LCH_k(\mathcal{A}_K(\Lambda)|_{Q_1}, \partial, \epsilon_{Q_1}) \oplus \\ & LCH_k(\mathcal{A}_K(\Lambda)|_{Q_2}, \partial, \epsilon_{Q_2}) \rightarrow LCH_k(\mathcal{A}_K(\Lambda), \partial, \epsilon) \rightarrow LCH_{k+1}(\mathcal{A}_K(\Lambda)|_N, \partial, \epsilon_N) \rightarrow \dots \end{aligned}$$

Where LCH_* denotes the linearized contact homology.

Next, let $\Lambda_+, \Lambda_- \subset J^1(M)$ be Legendrian submanifolds. We define an **exact Lagrangian cobordism** from Λ_+ to Λ_- to be an exact Lagrangian submanifold $L \subset J^1(M) \times \mathbb{R}$ such that:

- There exists $T > 0$ such that:

$$\mathcal{E}_+(L) = L \cap (J^1(M) \times (T, \infty)) = \Lambda_+ \times (T, \infty)$$

$$\mathcal{E}_-(L) = L \cap (J^1(M) \times (-\infty, -T)) = \Lambda_- \times (-\infty, -T)$$

- If $df = \beta|_L$, then f is constant on $\mathcal{E}_+(L)$ and on $\mathcal{E}_-(L)$.
- $L - (\mathcal{E}_+(L) \cup \mathcal{E}_-(L))$ is compact with boundary $\Lambda_+ \cup \overline{\Lambda_-}$.

We refer to $\mathcal{E}_+(L), \mathcal{E}_-(L)$ as the **positive and negative cones** of L . An exact Lagrangian cobordism L defines a homomorphism $\Phi_L : \mathcal{A}_K(\Lambda_+) \rightarrow \mathcal{A}_K(\Lambda_-)$, as discussed in section 2.4. We have an equivalent of our Sievert-van Kampen theorem for these cobordisms as well:

Theorem 1.3. *Let $L \subset J^1(M) \times \mathbb{R}$ be an exact Lagrangian cobordism from Λ_+ to Λ_- , and let $\hat{S} \subset M \times \mathbb{R}$ be a hypersurface that divides $M \times \mathbb{R}$ into two components \hat{R}_1 and \hat{R}_2 , such that $\hat{S} \cap (M \times \{t_0\})$ is also a hypersurface for any choice of t_0 , and such that \hat{S} does not cross any cusp edges of the Legendrian lift of L . Let $R_i^\pm = \hat{R}_i \cap \{t = \pm T\}$, $S^\pm = \hat{S} \cap \{t = \pm T\}$, let N^\pm be an arbitrarily small neighborhood of S^\pm , and let $Q_i^\pm = R_i^\pm \cup N^\pm$. Then L is exact Lagrangian isotopic to an exact Lagrangian cobordism L' from Λ'_+ to Λ'_- , where Λ'_\pm are Legendrian isotopic to Λ_\pm , and such that the image of the restriction of $\Phi_{L'}$ to $\mathcal{A}_K(\Lambda'_\pm)|_{Q_i^+}, \mathcal{A}_K(\Lambda'_\pm)|_{N^+}$ lies in $\mathcal{A}_K(\Lambda'_\pm)|_{Q_i^-}, \mathcal{A}_K(\Lambda'_\pm)|_{N^-}$.*

As applications of Theorems 1.1 and 1.3, we then prove that:

Theorem 1.4. *Let $\Lambda \subset J^1(M)$ be a Legendrian submanifold, and let $\hat{\Lambda} \subset J^1(M)$ be the product of a critical Legendrian ambient surgery on Λ . Then the differential graded algebras $\mathcal{A}(\Lambda)$ and $\mathcal{A}(\hat{\Lambda})$ are stable tame isomorphic to a pair of differential graded algebras A and \hat{A} , such that:*

- Let $\hat{x}_1, \dots, \hat{x}_m, \hat{d}$ be the generators of \hat{A} . Then the generators of A are x_1, \dots, x_m, d, c .
- For $x \neq c$, $\partial \hat{x} = \widehat{\partial x}$
- $\partial c = 1 + d$.

In section 2, we provide an overview of the concepts behind Legendrian Contact Homology and exact Lagrangian cobordisms. In section 3, we will use theorem 1.1 to prove theorem 1.4. In section 4, we will prove theorems 1.1, 1.2, and 1.3.

2 Overview of Legendrian Contact Homology

2.1 Topological Preliminaries

A **contact manifold** is a $(2n + 1)$ -dimensional manifold Y equipped with a hyperplane distribution ζ such that there exists no embedded $(2n)$ -dimensional submanifold that is everywhere tangent to ζ . A **contact form** α is a one-form in T^*Y such that $\zeta = \ker \alpha$. Although a contact form can always be found locally, a global contact form does not always exist. However, in our case we are limiting our interest to one-jet manifolds $Y = J^1(M) = T^*M \times \mathbb{R}$, where ζ is the kernel of the global contact form $\alpha = dz - \theta$, where θ is the tautological one-form of T^*M and z is the \mathbb{R} coordinate. In addition, we are particularly interested in the case $M = \mathbb{R}^n$, $J^1(M) = \mathbb{R}^{2n+1}$. In this subcase, we use x_1, \dots, x_n for coordinates on M and y_1, \dots, y_n to be the corresponding cotangent coordinates. Then:

$$\alpha = dz - \sum_{i=1}^n y_i dx_i$$

An n -dimensional submanifold $\Lambda \subset Y$ is called **Legendrian** if $T\Lambda \subset \zeta$. This is equivalent to $\alpha|_{\Lambda} = 0$.

A **symplectic manifold** is a $(2n)$ -dimensional manifold X equipped with a closed 2-form ω such that ω^n is a volume form. We are interested in symplectic manifolds in part because we will frequently be working in T^*M , which is equipped with the canonical symplectic form $\omega = d\theta$. Recall that an n -dimensional submanifold L of a symplectic manifold is called **Lagrangian** if $\omega|_L = 0$. If $\Lambda \subset J^1(M)$ is an embedded Legendrian submanifold, then $\pi_{\mathbb{C}}(\Lambda) \subset T^*M$ is an immersed Lagrangian submanifold, because we can define a function $z : \pi_{\mathbb{C}}(\Lambda) \rightarrow \mathbb{R}$ that is simply the z coordinate of that point in Λ , and:

$$(d\theta)|_{\pi_{\mathbb{C}}(\Lambda)} = d(dz) = 0$$

An **almost complex structure** on a $(2n)$ -dimensional manifold X is a smooth linear map $TX \rightarrow TX$ such that $J \circ J = -\text{Id}$. The standard example is the complex plane \mathbb{C} with real coordinates x, y and canonical contact form i , where:

$$i(\partial_x) = \partial_y$$

$$i(\partial_y) = -\partial_x$$

Given two manifolds with almost complex structures (X_1, J_1) and (X_2, J_2) , we say that a function $f : X_1 \rightarrow X_2$ is **pseudoholomorphic** if $f_* \circ J_1 = J_2 \circ f_*$.

Let (X, J, g, ω) be a $(2n)$ -dimensional manifold equipped with an almost complex structure J , a Riemannian metric g , and a symplectic form ω . We say that the triple (J, g, ω) is **tame** if g is complete and there exists constants r_0, C_1, C_2 such that:

- Every loop $\gamma \subset X$ contained in a ball $B_r(x)$ with $r \leq r_0$ bounds a disc in B of area less than $C_1(\text{length}(\gamma))^2$.

- $\|\omega_x\|_g \leq 1$ for all $x \in X$
- For every vector $V \in T_x X$, $|X|^2 \leq C_2 \omega(X, JX)$.

We can always equip T^*M with a tame triple (J, g, ω) ([1], Ch. 5, Sec. 4.1), and from now on will assume we have done so. Furthermore, the space of almost complex structures J on T^*M which is part of a tame triple is contractible ([10], Proposition 4.1.)

2.2 Definition of the DGA

We say that a Legendrian submanifold is **front generic** if it has the following properties:

- The base space projection $\pi_M : \Lambda \rightarrow M$ is an immersion outside of a codimension-1 submanifold Σ_1 .
- We define Σ_k inductively to be a codimension-1 subset of Σ_{k-1} , such that the map $\pi_M : (\Sigma_{k-1} - \Sigma_k) \rightarrow M$ is an immersion.
- At points $s \in \Sigma_1 - \Sigma_2$, $\pi_F : \Lambda \rightarrow M \times \mathbb{R}$ has a standard **cusp edge singularity**. That is, there exist coordinates u_1, \dots, u_n of Λ around s and coordinates x_1, \dots, x_n of M around $\pi_M(s)$ such that, if z is the fiber coordinate in $J^0(M) = M \times \mathbb{R}$, then $\pi_F(u) = (x_1(u), \dots, x_n(u), z(u))$, where:

$$\begin{aligned} x_1(u) &= \frac{1}{2}u_1^2 \\ x_j(u) &= u_j \text{ for } j = 2, \dots, n \\ z(u) &= \frac{1}{3}u_1^3 + \beta \frac{1}{2}u_1^2 + \alpha_2 u_2 + \dots + \alpha_n u_n \end{aligned}$$

Where $\alpha_2, \dots, \alpha_n, \beta$ are constants.

As discussed in [6], Section 2.2.1, any Legendrian submanifold Λ can be made front generic after an arbitrarily small Legendrian isotopy. We will therefore assume from here on that our Legendrian submanifolds are all front generic.

A **Reeb vector field** R_α is the vector field of $J^1(M)$ such that:

$$\alpha(R_\alpha) = 1$$

$$(d\alpha)(R_\alpha) = 0$$

For $J^1(M)$, $R_\alpha = \partial_z$. A **Reeb chord** of a Legendrian submanifold Λ is a trajectory of R_α that begins and ends on Λ . For $R_\alpha = \partial_z$, Reeb chords correspond precisely to self-intersections of $\pi_{\mathbb{C}}(\Lambda)$. These intersections will generically be transverse double points. We label the quadrants of each Reeb chord in $\pi_{\mathbb{C}}(\Lambda)$ with the signs shown in Figure 1.

We define $\mathcal{A}_K(\Lambda)$ to be the formal unital algebra over K , where $K = \mathbb{Z}_2$ or $\mathbb{Z}_2 H_1(\Lambda)$, generated by the Reeb chords of Λ - that is, elements of $\mathcal{A}_K(\Lambda)$ are

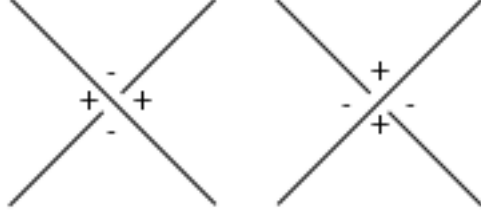


Figure 1: Labels of Quadrants of $\pi_{\mathbb{C}}(\Lambda)$

sentences made of words $kb_1...b_m$, where $k \in K$ and $b_1, ..., b_m$ are Reeb chords. We provide a differential to $\mathcal{A}_K(\Lambda)$ by counting rigid pseudoholomorphic disks in T^*M .

First, for every Reeb chord b of Λ , let b^+, b^- be the upper and lower end points of the Reeb chord. We choose a **capping path** for each chord b , which is a path $\gamma_b : [0, 1] \rightarrow \Lambda$ such that $\gamma(0) = b^+, \gamma(1) = b^-$. We define D_r to be the unit disk in \mathbb{C} with the canonical complex structure, with r punctures on its boundary labeled $q_0, ..., q_{r-1}$. Then, if $a, b_1, ..., b_m$ are Reeb chords and A is an element in $H_1(\Lambda)$, we define the **moduli space** $\mathcal{M}_A(a; b_1...b_m)$ to be the space of maps $u : (D_{m+1}, \partial D_{m+1}) \rightarrow (T^*M, \pi_{\mathbb{C}}(\Lambda))$ such that:

- u is pseudoholomorphic.
- As z limits to q_0 , $u(z)$ limits to a positive quadrant of a .
- As z limits to q_i , $i > 0$, $u(z)$ limits to a negative quadrant of b_i .
- The restriction of u to ∂D_{m+1} has a continuous lift $\tilde{u} : \partial D_{m+1} \rightarrow \Lambda \subset J^1(M)$.
- The homology class of $\tilde{u}(\partial D_{m+1} \cup \gamma_a \cup \gamma_{b_1} \cup ... \gamma_{b_m})$ equals $A \in H_1(\Lambda)$.

We define $\mathcal{M}(a; b_1...b_m)$ in the same fashion, except that we drop the requirement on the homology class.

These moduli spaces are open manifolds with boundary. We define the differential ∂ of $\mathcal{A}_K(\Lambda)$ by counting 0-dimensional moduli spaces. If $K = \mathbb{Z}_2$, then:

$$\partial a = \sum_{\dim \mathcal{M}(a; b_1...b_m)=0} (\#\mathcal{M}(a; b_1...b_m)) b_1...b_m$$

If $K = \mathbb{Z}_2 H_1(\Lambda)$, then:

$$\partial a = \sum_{\dim \mathcal{M}_A(a; b_1...b_m)=0} (\#\mathcal{M}_A(a; b_1...b_m)) A b_1...b_m$$

We extend these differentials to $\partial : \mathcal{A}_K(\Lambda) \rightarrow \mathcal{A}_K(\Lambda)$ using the linearity and the product rules:

$$\partial(x + y) = (\partial x) + (\partial y)$$

$$\partial(xy) = (\partial x)y + x(\partial y)$$

We say that a pseudoholomorphic disk in a 0-dimensional moduli space is **rigid**. Note that we could have zero negative punctures of a rigid pseudoholomorphic disk, in which case the disk contributes 1 to the differential. And:

Theorem 2.1. $\partial^2 = 0$, so the homology of $\mathcal{A}_K(\Lambda)$ is well-defined.

Proof: See [5], Proposition 2.6.

The differential graded algebra $\mathcal{A}_K(\Lambda)$ is *not* an invariant of the Legendrian isotopy class of Λ . However, its stable tame isomorphism class *is*. A **stabilization** of $\mathcal{A}_K(\Lambda)$, which we denote $S(\mathcal{A}_K(\Lambda))$, is a differential graded algebra whose generators consist of the generators of $\mathcal{A}_K(\Lambda)$, plus two generators b, c , and whose differential ∂_S is defined to be:

$$\partial_S x = \partial x \text{ if } x \in \mathcal{A}_K(\Lambda)$$

$$\partial_S b = c$$

$$\partial_S c = 0$$

An **elementary automorphism** of a differential graded algebra generated by a_1, \dots, a_n is an automorphism such that there exists some j so that $a_i \rightarrow a_i$ for $i \neq j$ and $a_j \rightarrow a_j + u$, where u is a word in $a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_n$. A **tame automorphism** is an automorphism that is the composition of a series of elementary automorphisms. A **tame isomorphism** between a differential graded algebra generated by a_1, \dots, a_n and a differential graded algebra generated by b_1, \dots, b_n is the composition of a tame automorphism with the isomorphism sending $a_1 \rightarrow b_1, \dots, a_n \rightarrow b_n$.

We say that two differential graded algebras $\mathcal{A}_K(\Lambda_1), \mathcal{A}_K(\Lambda_2)$ are **stable tame isomorphic** if they are tame isomorphic after some series of stabilizations. And:

Theorem 2.2. *The stable tame isomorphism class of $\mathcal{A}_K(\Lambda)$ is invariant under Legendrian isotopy.*

Proof: See [5], Proposition 2.7.

The homology of $\mathcal{A}_K(\Lambda)$ is invariant under stable tame isomorphism, and so is in turn an invariant of the Legendrian isotopy class. This homology is called the Legendrian Contact Homology of Λ .

When $\dim \Lambda = 1$, it is easy to find these pseudoholomorphic disks, because, thanks to the Riemann Mapping Theorem, they correspond to polygons in T^*M whose vertices are Reeb chords and whose edges lie on $\pi_{\mathbb{C}}(\Lambda)$. It is significantly harder to calculate the differential when $\dim \Lambda > 1$. Fortunately, as we discuss in the next subsection, [6] provides a technique for calculating the differential in a very broad special case.

2.3 Gradient Flow Trees

[6] provides a technique for calculating ∂ provided that either $\dim \Lambda = 1$ or 2 or the front projection π_F has no codimension-2 or higher singularities that are not cusp edge intersections. This is a large and very interesting class of Legendrian submanifolds. He does this using an object called a gradient flow tree.

Consider the front projection of a Legendrian submanifold. We can regard this as the graph of some collection of **height functions** $f_i : U_i \rightarrow \mathbb{R}, U_i \subset M$. A path $\gamma : [0, 1] \rightarrow M$ is a **flow line** of the height functions f_1, f_2 if $\gamma'(t) = -\nabla(f_1 - f_2)(\gamma(t))$. If $\gamma : [0, 1] \rightarrow M$ is a flow line, the **1-jet lift** of γ is an unordered pair $\{\tilde{\gamma}^1, \tilde{\gamma}^2\} : [0, 1] \rightarrow \Lambda \subset J^1(M)$ of continuous paths such that $\pi_F(\tilde{\gamma}_i(t)) = (\gamma(t), f_i(\gamma(t)))$. We define the **flow orientation** of $\tilde{\gamma}_1$ at a point $p \in M$ to be the unique lift of the vector $-\nabla(f_1 - f_2)(\pi_M(p)) \in T_{\pi_M(p)}M$ to $T_p\Lambda$.

Let Γ be a tree with finitely many edges. For $k \geq 2$, at each k -valent vertex, place a cyclic ordering on the edges. A map $F : \Gamma \rightarrow M$ is a **gradient flow tree** if it satisfies the following conditions:

- If e is an edge of Γ then the restriction of F to e is a flow line.
- If v is a k -valent vertex with ordered edges e_1, \dots, e_k , and $\tilde{\gamma}_j^1, \tilde{\gamma}_j^2$ are the 1-jet lifts of F restricted to e_j , then we require that $\pi_{\mathbb{C}}(\tilde{\gamma}_j^2(v)) = \pi_{\mathbb{C}}(\tilde{\gamma}_{j+1}^1(v))$, with the convention $\gamma_{k+1} = \gamma_1$. In addition, we require that the flow orientation of γ_j^2 at v is directed towards v if and only if the flow orientation of γ_{j+1}^1 is directed away.
- The Lagrangian projection of the 1-jet lifts of the edges of Γ fit together to form a closed oriented curve $\tilde{\Gamma}$ in $\pi_{\mathbb{C}}(\Lambda)$.

Let $\mathcal{T}(a; b_1 \dots b_m)$ be the space of gradient flow trees with a positive puncture at a and negative punctures at b_1, \dots, b_m . Let $\tilde{\Gamma}$ be the lift of Γ to Λ , and let $\mathcal{T}_A(a; b_1 \dots b_m)$ be the restriction of $\mathcal{T}(a; b_1 \dots b_m)$ to trees such that the homology class of $\tilde{\Gamma} \cup \gamma_a \cup \gamma_{b_1} \cup \dots \cup \gamma_{b_m}$ equals A in $H_1(\Lambda)$.

Theorem 2.3. *If $\dim \Lambda = 1$ or 2 or if the only singularities of π_F are cusp edges, then there exists an almost complex structure such that there is a bijection between $\mathcal{T}(a; b_1 \dots b_m)$ and $\mathcal{M}(a; b_1 \dots b_m)$, and between $\mathcal{T}_A(a; b_1 \dots b_m)$ and $\mathcal{M}_A(a; b_1 \dots b_m)$, if the disks are rigid.*

Proof: See [6], Theorem 1.1.

For a Reeb chord p , we will use $I(p)$ to denote the Morse index of the height difference function at p . If $I(p) = 0$ we say p is a **minimum**, and if $I(p) = \dim \Lambda$ we say p is a **maximum**.

The vertices of a gradient flow tree may correspond to Reeb chords or they may correspond to other joinings of flow lines. We label the following kinds of non-chord vertices:

- **End:** One-valent vertex where a flow line between two sheets meets a cusp edge of that sheet.
- **Switches:** Two-valent vertex where a flow line between sheets i and j , where i lies above j , meets a cusp edge between sheets i and k (or k and j), and leaves as a flow line between sheets k and j (or i and k).
- **Y^0 vertices:** Three-valent vertex where a flow line between sheets i and j splits into two flow lines, one between sheets i and k and one between sheets k and j , where the z coordinate of sheet k lies between the z coordinates of sheets i and j .
- **Y^1 vertices:** Three-valent vertex where a flow line between sheets i and j meets a cusp edge between sheets k and l , where i lies above k and l lies above j , and leaves as two flow lines, one between sheets i and l and one between sheets k and j .

Theorem 2.4. *Let Γ be a gradient flow tree with positive Reeb chords p_1, \dots, p_m , negative Reeb chords q_1, \dots, q_r , and non-chord vertices v_1, \dots, v_s . The dimension of the moduli space containing Γ is then:*

$$\dim \Gamma = (n - 3) + \sum_{i=1}^m (I(p_i) - (n - 1)) - \sum_{j=1}^r (I(q_j) - 1) + \sum_{k=1}^s \mu(v_k) \quad (2.1)$$

Where $I(x)$ is the Morse index of x if x is a Reeb chord, $I(x) = n + 1$ if x is a positive special puncture, $I(x) = -1$ if x is a negative special puncture, $\mu(v_k) = 1$ if v_k is an end, 0 if v_k is a Y^0 vertex, and -1 if v_k is a switch or Y^1 vertex.

Proof: See [6], Prop. 3.14.

Theorem 2.5. *Any vertex of a rigid gradient flow tree with one positive puncture must be a one-valent vertex at a positive or negative Reeb chord; a two-valent vertex at a positive minimum or a negative maximum; a cusp edge; a switch; a Y^0 vertex; or a Y^1 vertex.*

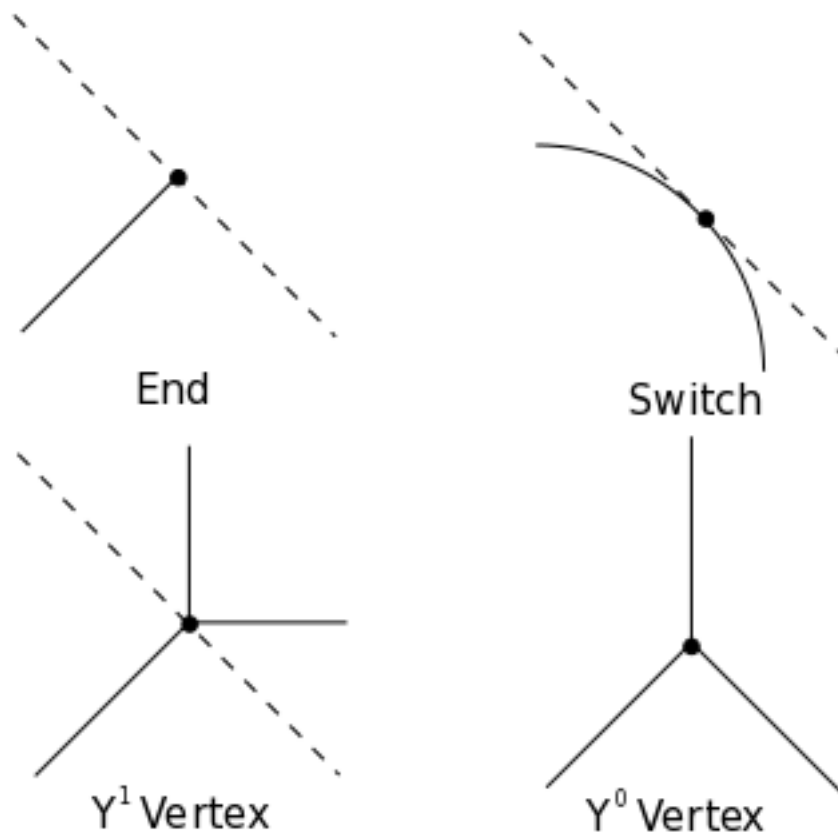


Figure 2: Non-Chord Vertices of Gradient Flow Trees
(Solid lines are gradient flow lines, dotted lines are cusp edges)

Proof: See [6], Lemma 3.7.

We will also make use of objects called **partial flow trees**, which are gradient flow trees that have one or more **special punctures**, which are 1-valent vertices that are neither cusp edges nor Reeb chords. A special puncture may be considered to be either positive or negative, depending on the orientation of the 1-jet lift. If v is a special puncture, then we treat $I(v) = n + 1$ if v is positive, $I(v) = -1$ if v is negative. Critically, given any gradient flow tree Γ , and any point p in a flow line (not a vertex) of that tree, we can separate Γ into a pair of partial flow trees, Γ_1 and Γ_2 , by breaking Γ at p . Then:

$$\dim \Gamma = \dim \Gamma_1 + \dim \Gamma_2 - (n + 1)$$

The advantage of gradient flow trees is that they reduce finding the rigid pseudoholomorphic disks from an infinite-dimensional problem to a finite-dimensional problem. Although we cannot always use them, given the requirements on π_F , they are a very valuable tool when they are useable. The essence of the proof of theorems 1.1 through 1.5 will be to show that, although we cannot prove that pseudoholomorphic disks are globally equivalent to gradient flow trees if π_F has higher-codimension singularities, we can show they are *locally* equivalent.

2.4 Augmentations and Linearized Contact Homology

The Legendrian Contact Homology of a Legendrian submanifold $\Lambda \subset J^1(M)$ is, generally, infinite-dimensional, and often quite difficult to work with. We therefore define algebraic invariants of the Legendrian Contact Homology which will be finite-dimensional.

First, we split the differential ∂ into components $\partial_0, \partial_1, \partial_2$, etc., where ∂_k denotes the restriction of ∂ to words of length k . Since $\partial \circ \partial = 0$, we obtain:

$$\partial_0 \circ \partial_0 = 0$$

$$(\partial_0 \circ \partial_1) + (\partial_1 \circ \partial_0) = 0$$

$$(\partial_0 \circ \partial_2) + (\partial_1 \circ \partial_1) + (\partial_2 \circ \partial_0) = 0$$

Etc. This means that, if $\partial_0 = 0$, then $\partial_1 \circ \partial_1 = 0$, which means that we can define a **linearized chain complex** given by the restriction of $\mathcal{A}(\Lambda)$ to words of unit length, and equipped with the differential ∂_1 . The homology of this algebra will then be finite-dimensional. Unfortunately, $\partial_0 \neq 0$ in general. However, $\mathcal{A}(\Lambda)$ may be stably tame isomorphic to a DGA where ∂_0 *does* equal zero.

Finding such an isomorphism is equivalent to finding a graded algebra map $\epsilon : \mathcal{A}(\Lambda) \rightarrow \mathbb{Z}_2$ such that $\epsilon(1) = 1$ and $\epsilon \circ \partial = 0$. We call such a map an **augmentation**. Given an augmentation, we define a DGA map by $H_\epsilon(x) = x + \epsilon(x)$ if x is a Reeb chord, and extend to products multiplicatively - that is:

$$H_\epsilon(xy) = H_\epsilon(x)H_\epsilon(y)$$

H_ϵ is a stable tame isomorphism to the differential graded algebra with the same generators as $\mathcal{A}(\Lambda)$, but with differential $\partial_\epsilon = H_\epsilon \circ \partial \circ H_\epsilon^{-1}$. $(\partial_\epsilon)_0$ will always equal zero, so we can define a linearized chain complex for this augmented algebra. The homology of this complex is called the **linearized contact homology**, or $LCH_*(\mathcal{A}(\Lambda), \partial, \epsilon)$. Since ϵ is not necessarily unique, a given Legendrian may have multiple linearized contact homologies, and two different maps ϵ may produce the same linearized contact homology. However:

Theorem 2.6. *The set of linearized contact homologies of Λ is an invariant of the stable tame isomorphism class of $\mathcal{A}(\Lambda)$, and therefore is an invariant of the Legendrian isotopy class of Λ .*

Proof: See [2], Theorem 5.2.

2.5 Cobordism Maps

Let $\mathbb{R}^+ = (0, \infty)$. The symplectization $(J^1(M) \times \mathbb{R}, d(e^t \alpha))$ is symplectomorphic to the cotangent bundle with canonical symplectic form $(T^*(M \times \mathbb{R}^+), \omega)$ by the map:

$$F(x_1, \dots, x_n, y_1, \dots, y_n, z, t) = (x_1, \dots, x_n, e^t, e^t y_1, \dots, e^t y_n, z)$$

Although this is not an *exact* symplectomorphism, the difference between the primitive of the symplectic form of $J^1(M) \times \mathbb{R}$ and the tautological one-form of $T^*(M \times \mathbb{R}^+)$ is exact:

$$F^* \left(- \sum_{i=1}^n y_i dx_i - z dt \right) = - \sum_{i=1}^n e^t y_i dx_i - e^t z dt = e^t \alpha - d(e^t z)$$

Therefore, if $(e^t \alpha)|_L = df$, then $\theta|_{F(L)} = d(f + tz)$, where θ is the tautological one-form of $T^*(M \times \mathbb{R}^+)$. So $F(L)$ is still an exact Lagrangian cobordism. We will use L to refer to $F(L)$ from here on, and work exclusively in $T^*(M \times \mathbb{R}^+)$. We prefer this environment because it is the native environment for our pseudoholomorphic disks.

An exact Lagrangian cobordism L from Λ_+ to Λ_- induces a DGA map $\Phi_L : \mathcal{A}(\Lambda_+) \rightarrow \mathcal{A}(\Lambda_-)$, given by counting rigid pseudoholomorphic disks whose punctures limit to the Reeb chords at infinity. That is, if a is a Reeb chord in Λ_+ and b_1, \dots, b_m are Reeb chords in Λ_- , we define the moduli space $\widetilde{\mathcal{M}}(a; b_1 \dots b_m)$ to be the space of maps $u : (D_{m+1}, \partial D_{m+1}) \rightarrow (T^*(M \times \mathbb{R}^+), L)$ such that:

- u is pseudoholomorphic, that is, $du + J \circ du \circ J_{\mathbb{C}} = 0$, where $J_{\mathbb{C}}$ is the almost complex structure on the domain of u .
- $u(p)$ limits to $a \times \{\infty\}$ as $p \rightarrow q_0$, where a is a Reeb chord of Λ_+ .
- $u(p)$ limits to $b_i \times \{-\infty\}$ as $p \rightarrow q_i$, where b_i is a Reeb chord of Λ_- .

We then define Φ_L by:

$$\Phi_L(a) = \sum_{\dim \widetilde{\mathcal{M}}(a; b_1 \dots b_m) = 0} \left(\# \widetilde{\mathcal{M}}(a; b_1 \dots b_m) \right) b_1 \dots b_m$$

Let ∂_{\pm} denote the differential of Λ_{\pm} . Then:

Theorem 2.7. $\Phi_L \circ \partial_+ = \partial_- \circ \Phi_L$

Proof: See [7], Theorem 1.2.

Theorem 2.8. *If L_1 and L_2 are exact Lagrangian isotopic, then Φ_{L_1} is chain homotopic to Φ_{L_2} .*

Proof: See [7], Theorem 1.3.

Note that, given an exact Lagrangian cobordism L_{12} from Λ_1 to Λ_2 , and a second exact Lagrangian cobordism L_{23} from Λ_2 to Λ_3 , we can form a new exact Lagrangian cobordism L_{13} from Λ_1 to Λ_3 by concatenating L_{12} and L_{23} : we delete the negative cone of L_{12} and positive cone of L_{23} and glue them together. Then:

Theorem 2.9. $\Phi_{L_{13}} = \Phi_{L_{23}} \circ \Phi_{L_{12}}$

Proof: See [7], Theorem 1.2.

2.6 Lifting Cobordisms

This section is derived from [7], Section 2, with minor modifications to adapt the concept to higher dimensions - principally adding extra coordinates as appropriate.

Recall that, according to the definition of an exact Lagrangian cobordism $L \subset T^*(M \times \mathbb{R}^+)$, there exists a function $f : L \rightarrow \mathbb{R}$ such that $\beta|_L = d(f + tz)$. We can lift $L \subset T^*(M \times \mathbb{R}^+)$ to $\hat{L} \subset J^1(M \times \mathbb{R}^+)$, with \hat{L} unique up to \mathbb{R} -translation, by mapping $p \in L$ to $(p, f(p) + t(p)z(p)) \in \hat{L}$. Let $\hat{\alpha}$ be the canonical contact form of $J^1(M \times \mathbb{R}^+)$; then:

$$\hat{\alpha}|_{\hat{L}} = d(f + tz) - \theta = \theta - \theta = 0$$

So \hat{L} is Legendrian, though it is definitely *not* compact. However, as we will see in section 4, we do not *need* compactness. We will prove convergence of the boundary of pseudoholomorphic curves to gradient flow lines in Λ , where Λ may denote a compact Legendrian submanifold of $J^1(M)$ or it may denote the

Legendrian lift $\hat{L} \subset J^1(M \times \mathbb{R})^+$ of an exact Lagrangian cobordism, and where M may mean M or may mean $M \times \mathbb{R}^+$.

This does create some confusion between the z coordinate of Λ_\pm and the new z coordinate of \hat{L} given by integrating $d(f + tz)$. In section 4, when we refer to the z coordinate of Λ , we will always mean the \mathbb{R} coordinate of the one-jet space, whether that one-jet space happens to be $J^1(M)$ or $J^1(M \times \mathbb{R}^+)$.

3 Critical Legendrian Ambient Surgery

Legendrian Ambient Surgery is defined in [12], Sec. 3, as an analogue of Morse surgery for Legendrians. We begin with the following data: let $\Lambda \subset J^1(M)$ be an n -dimensional Legendrian submanifold, and let $k < n$. Let $S \subset \Lambda$ be an embedded k -sphere with a choice of framing F of its normal bundle $NS \subset T\Lambda$. We define the **symplectic normal bundle of NS** over S to be:

$$NS^{(d\alpha)} = \{v \in \text{Ker } \alpha \mid (d\alpha)(v, NS) = \{0\}\}$$

Let $D_S \subset J^1(M)$ be an isotropic embedded $(k+1)$ -disk that satisfies the following properties:

- $\partial D_S = S$
- $\text{Int } D_S \subset J^1(M) - \Lambda$
- Any outward normal vector field to D_S is nowhere tangent to Λ
- For any vector field H in NS such that $(d\alpha)(G, H) > 0$ for any vector G that is outward normal to D_S , we require that the frame given by adjoining H to the Lagrangian frame of $(TD_S)^{d\alpha}|_S$ is homotopic to F .

[12] uses the above data to construct a standard model of a neighborhood of $S \subset \Lambda$, then performs the surgery on this standard model, obtaining a surgered Legendrian submanifold $\hat{\Lambda}$. He calculates the change in $\mathcal{A}(\Lambda)$ for $k < n-1$, but is unable to calculate the change for $k = n-1$. Using pinching techniques, we can calculate the change in the algebra for $k = n-1$, using a modified version of the standard model, which is Legendrian isotopic to [Ri]'s version.

3.1 Preliminary Morse Lemmas

We begin with some preliminary Morse lemmas, which will be used in constructing our local neighborhood model. First, we define:

$$\mathcal{F}(f_1, \delta) = \{f_2 : M \rightarrow \mathbb{R} \mid f_2 \text{ is Morse, } |f_1 - f_2|_{C^1} < \delta\}$$

We then have the following Morse lemmas:

Lemma 3.1. *Let $f_1 : M \rightarrow \mathbb{R}$ be a Morse function, where M is a compact manifold. For any $\epsilon > 0$, there exists $\delta > 0$ such that, for any $f_2 \in \mathcal{F}(f_1, \delta)$, there is a bijection between the critical points of f_1 and f_2 , the ascending manifold of every critical point of f_2 lies within ϵ of the ascending manifold for the corresponding critical point of f_1 , and the descending manifold of every critical point of f_2 lies within ϵ of the descending manifold of the corresponding critical point of f_1 .*

Proof: See Appendix B.

Lemma 3.2. *Let $f : M \rightarrow \mathbb{R}$ be a Morse function, where M is a closed manifold. Let Q be a compact codimension-0 subset of M that includes no critical points of f . Then for any $\epsilon > 0$ there exists $\delta > 0$ such that for any critical point q and any points $p_1, p_2 \in Q$, if:*

$$d(p_1, p_2) < \delta, \text{ and}$$

$$p_1, p_2 \text{ lie in the same component of } \mathcal{A}_f(q)$$

Then:

$$d(\mathcal{D}_f(p_1), \mathcal{D}_f(p_2)) < \epsilon$$

And, for any $\epsilon > 0$, there exists $\delta > 0$ such that, for any points $p_1, p_2 \in Q$, if

$$d(p_1, p_2) < \delta, \text{ and}$$

$$p_1, p_2 \text{ lie in the same component of } \mathcal{D}_f(q)$$

Then:

$$d(\mathcal{A}_f(p_1), \mathcal{A}_f(p_2)) < \epsilon$$

Proof: See Appendix B.

Lemma 3.3. *Let $f_1 : M \rightarrow \mathbb{R}$ be a Morse function, where M is a compact manifold. For any choice of $\epsilon > 0$ and any compact codimension-0 submanifold $Q \subset M$ that lies in the descending manifold of maxima of f_1 and contains no critical points of f_1 , there exists δ such that, for any generic choice of $f_2 \in \mathcal{F}(f_1, \delta)$ and any $p \in Q$, the ascending manifold of p for ∇f_2 will lie within an ϵ -neighborhood of the ascending manifold of p for ∇f_1 .*

Proof: See Appendix B.

Lemma 3.4. *Let $\Lambda \subset J^1(M)$ be a front-generic Legendrian submanifold whose front projection is defined by sheet functions $f_1 : U_1 \rightarrow \mathbb{R}, \dots, f_m : U_m \rightarrow \mathbb{R}$, where $U_1, \dots, U_m \subset M$. Let $\hat{\Lambda} \subset J^1(M)$ be a second front-generic Legendrian submanifold whose front projection is defined by sheet functions $\hat{f}_1 : U_1 \rightarrow \mathbb{R}, \dots, \hat{f}_m : U_m \rightarrow \mathbb{R}$. Then, for any choice of $\epsilon > 0$, there exists $\delta > 0$ such that, if:*

$$\left\| f_i - \hat{f}_i \right\|_{C^1} < \delta \text{ for all } i$$

Then, there exists a bijection between the rigid gradient flow trees Λ with one positive Reeb chord and the rigid gradient flow trees of $\hat{\Lambda}$ with one positive Reeb chord, such that a tree Γ of Λ shares the same Reeb chords with its corresponding tree $\hat{\Gamma}$ of $\hat{\Lambda}$, and such that the projection of $\hat{\Gamma}$ to M lies within an ϵ -neighborhood of the projection of Γ to M .

Proof: See Appendix B.

3.2 Setup

We will use ordinary letters x to denote Reeb chords prior to the surgery, and we will use a superscript \hat{x} to denote Reeb chords after the surgery. We will use a superscript \bar{x} to denote Reeb chords prior to the perturbation (see below for an explanation of the perturbation). Since our surgery sphere S lies on Σ_1 , we can assume without loss of generality that, after a Legendrian isotopy, D_S is arbitrarily small, because we can contract it in the front projection through any intervening sheets. We may therefore further assume that, after a Legendrian isotopy, all sheets of the front projection $\pi_F(\Lambda)$ that lie over a neighborhood $U \subset M$ of $\pi_M(D_S)$ are flat hypersurfaces that do not cross $\pi_F(D_S)$ or each other.

Label the sheets in $\pi_F(D_S)$ with (possibly negative) integers by z order, so that sheet 0 and 1 are the sheets containing S . Pick a coordinate patch of M so that U corresponds to the n -disk centered at 0 of radius 4. This gives us a natural projection $\pi_S : (U - \{0\}) \rightarrow S^n$. We will use $D_r \subset M$ to denote $\{|x| \leq r\}$. We perform a Legendrian isotopy so that, over D_3 , the front projection of sheet i is the graph of a sheet function f_i which will be specified below. We will use \hat{f}_i to denote the sheet functions after the surgery. Note that, for $i \neq 0, 1$, $\hat{f}_i = f_i$.

We begin by defining a preliminary function $\tilde{f}(x)$:

- $\tilde{f}(x)$ has a maximum at $p = (2, 0, \dots, 0)$, with the value $\tilde{f}(p) = 2$.
- Let K be a 0.01-neighborhood of p . \tilde{f} has no other critical points in K .
- Over $D_{2.51} - D_{2.49}$, \tilde{f} has the value:

$$\tilde{f}(x) = ((|x| - 2.5)^2 + 0.01) + 0.0001s(\pi_S(x))$$

where π_S is the projection to S^n and $s : S^n \rightarrow [0, 1]$ is a function that has a maximum at the north pole, a minimum at the south pole, and no other critical points.

- Over $D_{2.48} - K$, \tilde{f} has the value:

$$\tilde{f}(x) = 2 \left(\frac{l(x) - |p - x|}{l(x)} \right) + 0.01 \left(\frac{|p - x|}{l(x)} \right)$$

Where $l(x)$ denotes the length of the straight line from p to $\partial D_{2.5}$ through x .

- \tilde{f} has no other critical points in D_3 .

Define:

$$M_i = \begin{cases} 2^i & i > 1 \\ 0 & i = 1 \\ 0 & i = 0 \\ -2^{-i} & i < 0 \end{cases}$$

Then define further preliminary functions $\tilde{f}_i(x) = M_i \tilde{f}(x)$. (It is intentional that $\tilde{f}_1 = \tilde{f}_0 = 0$ at this stage.) We will use **formally rigid** to denote the gradient flow trees on these height difference functions, excluding $\tilde{f}_i - \tilde{f}_j$, that have formal dimension zero (they are not actually rigid because \tilde{f}_i are not generic). Pick an arbitrarily small $\epsilon > 0$, and let V be an ϵ -neighborhood of the image in the base space M of the formally rigid gradient flow trees of all of the height difference functions $\tilde{f}_i - \tilde{f}_j, i > j$, excluding $\tilde{f}_1 - \tilde{f}_0$. Note that V includes, in particular, the gradient flow line from a_j^i to b_j^i , which coincide for all i, j ; this fact will be important later.

Find a disk $D' \subset D_{2.4} - V$ of radius ρ , choosing ρ to be arbitrarily small. In particular, choose ρ to be small enough that we can approximate the height functions $\tilde{f}_i, i \neq 0, 1$ linearly over D' . Let η be the infimum of $|\nabla \tilde{f}_i|, i \neq 0, 1$ over D' . Let $\delta_1 > 0$ be a quantity small enough that:

$$\delta_1 \leq \frac{\eta}{2(1+m)\rho^{3/2}}$$

Where m is the supremum of $|\nabla \tilde{f}|$.

By Lemma 3.4, we can find $\delta_2 > 0$ such that, if we perturb our height functions by an amount less than δ_2 , the image of the rigid gradient flow trees of the perturbed height functions will be within ϵ of the images of the rigid gradient flow trees of the height functions \tilde{f}_i , excepting the undefined flow trees on $\tilde{f}_1 - \tilde{f}_0$. Let δ be the minimum of δ_1, δ_2 .

For $i \neq 0, 1$, define $\bar{f}_i = \tilde{f}_i$, and define $\bar{f}_1 = \frac{1}{3m} \delta \tilde{f}, \bar{f}_0 = -\frac{1}{3m} \delta \tilde{f}$. Observe that \bar{f}_1, \bar{f}_0 are $\frac{1}{3}\delta$ -small perturbations in the C^1 metric.

Define \hat{f}_i to be generic perturbations of \bar{f}_i that are $\frac{1}{3}\delta$ -small in the C^1 -metric. The height difference functions $\hat{f}_i - \hat{f}_j$ have three clusters of critical points: maxima around p , which we denote by \hat{a} ; index- $(n-1)$ critical points around the north pole of $\partial(D_{2.5})$, which we denote by \hat{b} ; and minima around the south pole of $\partial(D_{2.5})$, which we denote by \hat{d} . We will use x_j^i to denote a critical point of type x for the height difference function between sheets i and j . A cross-section of D_3 is shown in Figure 3.

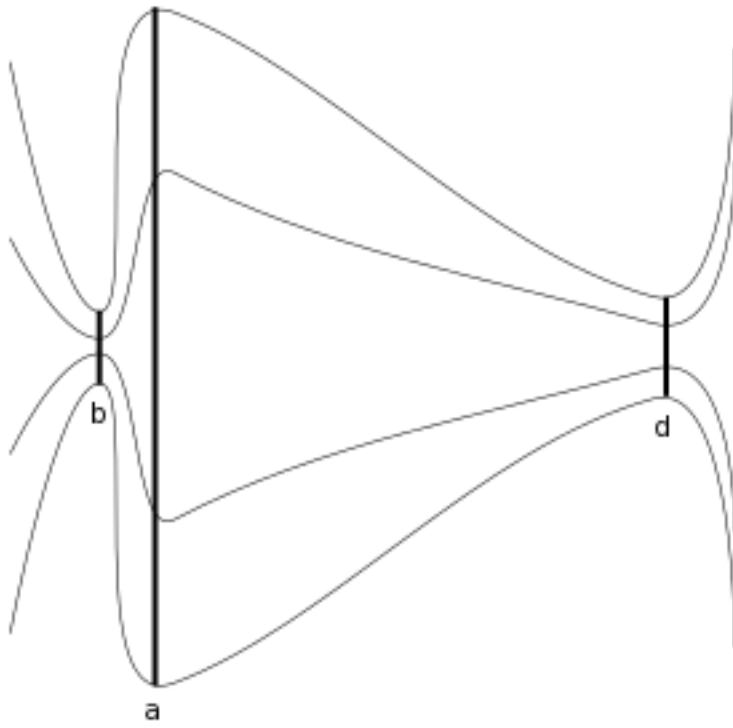


Figure 3: Cross-section of D_3

Let $\bar{a}, \bar{b}, \bar{d}$ denote the corresponding critical points of the height difference functions $\bar{f}_i - \bar{f}_j$ (we will use these in the following subsection).

Observe that, by construction, the rigid gradient flow trees of $\bar{f}_1 - \bar{f}_0$ coincide with the rigid gradient flow trees of $\bar{f}_{i+1} - \bar{f}_i, i \neq 0$. Therefore, the rigid gradient flow trees of \hat{f}_i lie within V for all i , and thus avoid D' .

Let q be the center of D' , and let $\pi_1 : (D' - \{q\}) \rightarrow \partial D'$ be the natural projection. Define $h_i(x) = f_i(\pi_1(x))$. Define the set:

$$D'_r = \{x \in D' \mid |x - q| \leq r\}$$

We define f_0^1, f_1^1 as follows:

- Outside of D'_ρ , $f_i^1 = \hat{f}_i$.
- Within $D'_{0.9\rho} - D'_{0.8\rho}$, f_i^1 has the form:

$$f_1^1(x) = \frac{h_1(x)}{(0.1\rho)^{3/2}} (|x - q| - 0.8\rho)^{3/2}$$

$$f_0^1(x) = \frac{h_0(x)}{(0.1\rho)^{3/2}} (|x - q| - 0.8\rho)^{3/2}$$

- f_0^1, f_1^1 are not defined inside $D'_{0.8\rho}$.
- Within $D'_\rho - D'_{0.9\rho}$:

$$|\nabla f_1^1|, |\nabla f_0^1| < \frac{1}{2}\eta$$

Where η is the infimum of $|\nabla f_i|, i \neq 0, 1$, over D' .

We define f_0, f_1 to be generic $\frac{1}{3}\delta$ -small perturbations of \bar{f}_1, \bar{f}_0 , and for $i \neq 0, 1$ we define $f_i = \hat{f}_i$.

Then, prior to the Legendrian ambient surgery, outside of D' we will have Reeb chords a, b, d corresponding to the chords $\hat{a}, \hat{b}, \hat{d}$. Within D' , we will have an index-1 Reeb chord c between sheets 0 and 1. However, observe that, for $i = 0, 1$, over D' :

$$\nabla f_i = \nabla h_i (|x - q| - 0.8\rho)^{3/2} + \frac{3}{2} h_i (|x - q| - 0.8\rho)^{1/2} \nabla |x - q|$$

$$|\nabla f_i| \leq |\nabla h_i| (0.1\rho)^{3/2} + \frac{3}{2} |h_i| (0.1\rho)^{1/2} \rho$$

Since h_i is equal to \hat{f}_i on the boundary, we can bound $|h_i|, |\nabla h_i|$ by $|\hat{f}_i|, |\nabla \hat{f}_i|$:

$$|\nabla f_i| \leq |\nabla \hat{f}_i| (0.1\rho)^{3/2} + \frac{3}{2} |\hat{f}_i| (0.1\rho)^{1/2} \rho$$

$$|\nabla f_i| \leq \delta(1 + m)\rho^{3/2} \leq \frac{1}{2}\eta$$

Where η is strictly less than the supremum of $|\nabla f_i|, i \neq 0, 1$ over D' . Therefore, no other Reeb chords are generated besides c . An "overhead" view of $\pi_F(\Lambda)$ will have the form shown in Figure 4.

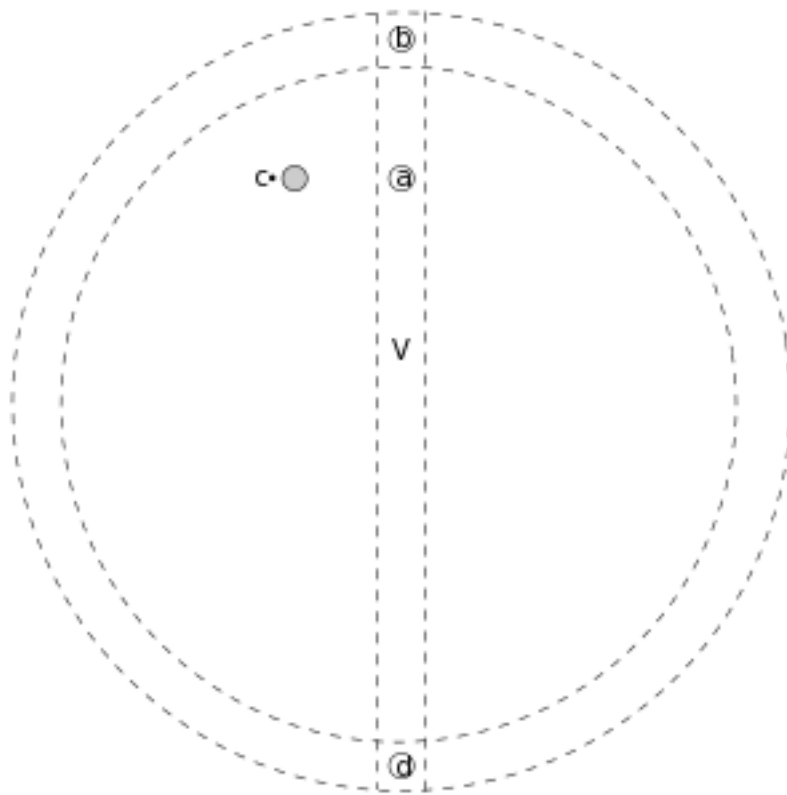


Figure 4: Overhead View of D_3

3.3 DGA Calculations:

We will do this in several steps. First of all:

Lemma 3.5. *If δ, ϵ, ρ are small enough, then there exists no partial flow tree Γ in Λ or $\hat{\Lambda}$ with a positive special puncture over D' and a negative puncture at an a or \hat{a} Reeb chord.*

Proof: Let A_1 be the union of the ascending manifolds of every Reeb chord a_j^i for every pair of height difference functions $-\nabla(f_k - f_l)$; let A_2 be the union of the ascending manifolds of A_1 for every pair of height difference functions $-\nabla(f_k - f_l)$; let A_3 be the union of the ascending manifolds of A_2 ; etc. Let m be the total number of sheets of Λ . We claim that A_1, \dots, A_m are disjoint from D' if ϵ, ρ are small enough.

Recall that \bar{f}_i denotes the sheet height functions before they are genericized. Let $U(a)$ be a Morse neighborhood of \bar{a} for $-\nabla(\bar{f}_i - \bar{f}_j)$. Recall that \bar{a} denotes the critical point of the height difference functions $\bar{f}_i - \bar{f}_j$ corresponding to the cluster of critical points a, \hat{a} , where \bar{f} are the height functions before the perturbation. (Since these functions are all equal to each other multiplied by a constant, they will have a common Morse neighborhood, except that the functions will take the form $\bar{f}_i(x) - \bar{f}_j(x) = \bar{f}_i(0) - \bar{f}_j(0) - m_i(x_1^2 + \dots + x_n^2)$ instead of $\bar{f}_i(0) - \bar{f}_j(0) - (x_1^2 + \dots + x_n^2)$.) If δ is small enough, the critical points a_j^i, \hat{a}_j^i will lie in $U(a)$. Define an outward-pointing radial vector field of $U(a)$:

$$R = x_1 \partial_{x_1} + \dots + x_n \partial_{x_n}$$

And observe $-\nabla(\bar{f}_i - \bar{f}_j) \cdot R > 0$ on $\partial U(a)$ for all i, j , so all gradient flows of the height difference functions $-(\bar{f}_i - \bar{f}_j)$ are leaving $U(a)$. Therefore, if δ is small enough, then $-\nabla(f_i - f_j) \cdot R > 0, -\nabla(\hat{f}_i - \hat{f}_j) \cdot R > 0$ on $\partial U(a)$ for all i, j , so all gradient flows of $-(f_i - f_j) = -(\hat{f}_i - \hat{f}_j)$ on $\partial U(a)$ are also leaving.

What this means is that, for every $p \in \partial U(a)$ and every height difference function $-(f_i - f_j) = -(\hat{f}_i - \hat{f}_j)$, the ascending manifold of p for that height difference function will lie inside $U(a)$. Therefore, if δ is small enough, the ascending manifold of every point $p \in U(a)$ for every choice of height difference function $-(f_i - f_j) = -(\hat{f}_i - \hat{f}_j)$ will lie in $U(a)$. Therefore, $A_1, \dots, A_m \subset U(a)$. Since $U(a)$ is disjoint from D' if δ, ϵ, ρ are small enough, we conclude that A_1, \dots, A_m is disjoint from D' .

Why does this matter? Well, consider a partial flow tree Γ in Λ or $\hat{\Lambda}$ with a positive special puncture over D' and a negative puncture at an a or \hat{a} chord. To simplify notation, assume without loss of generality this tree is in Λ . Let $U(a)'$ be an arbitrarily small open neighborhood of $U(a)$ that is also disjoint from D' . Let Γ' be the subtree of Γ that is obtained by restricting Γ to $U(a)'$ and deleting the components of the restricted tree that do not contain our Reeb chord at a_j^i . Γ' will then have a positive special puncture q on $\partial(U(a)')$, a negative puncture a_j^i , and possibly some Y^0 vertices and/or negative special

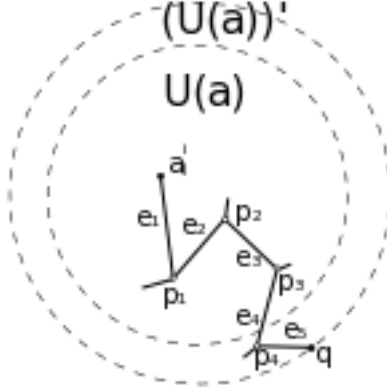


Figure 5: Labels of Edges in $U(a)$

punctures on $\partial(U(a))'$. Consider the sequence of edges e_1, \dots, e_k in Γ' between a_j^i and q : a_j^i is a boundary point of e_1 , define p_1 to be the joint boundary point of e_1 and e_2 , p_2 to be the joint boundary point of e_2 and e_3 , and so on, up to $p_k = q$ is a boundary point of e_k , as shown in Figure 5. Note that, since each edge e_i must lie between fewer sheets than the edge e_{i+1} , we can conclude that $k \leq m$.

Then $p_1 \in A_1, p_2 \in A_2, \dots, p_k \in A_k$, so q lies in A_k . But $A_k \subset U(a)$, and $q \notin U(a)$, which is a contradiction. Therefore Γ cannot have a negative puncture at an a chord.

Lemma 3.6. *If δ, ϵ, ρ are small enough, then there exists no partial flow tree Γ in Λ or $\hat{\Lambda}$ with a positive special puncture over D' and a negative puncture at a b or \hat{b} Reeb chord.*

Proof: Let B_1 denote the union of the ascending manifolds of b_j^i for $f_i - f_j = \hat{f}_i - \hat{f}_j$ for all i, j . Let B_2 denote the union of the ascending manifolds of B_1 for $f_k - f_l = \hat{f}_k - \hat{f}_l$ for all k, l ; let B_3 denote the union of the ascending manifolds of B_2 ; and so on, up to B_m , where m is the number of sheets of $\pi_F(\Lambda)$.

Let $U(b)$ be a Morse neighborhood of \bar{b} , and assume δ is small enough that $b_j^i = \hat{b}_j^i \in U(b)$ for all i, j . As in equation B.1 in the proof of Lemma 3.1, define S to be a ϵ_0 -neighborhood of the ascending manifolds of \bar{b} for $-\nabla(\bar{f}_k - \bar{f}_l)$:

$$S = \{(x_1, \dots, x_n) \in U(b) | x_2^2 + \dots + x_n^2 \leq \epsilon_0^2\}$$

Consider ∂S as the union of subsets V, W , as in equation B.2

$$V = \{(x_1, \dots, x_n) \in U(b) | x_2^2 + \dots + x_n^2 = \epsilon_0^2\}$$

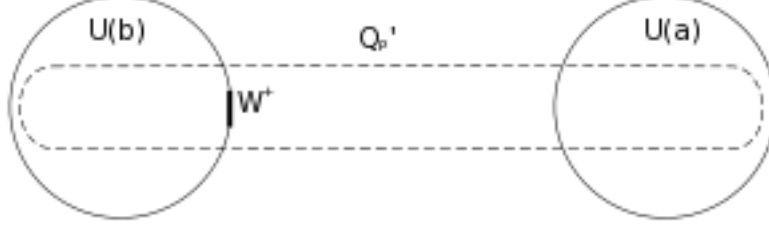


Figure 6: Diagram of $Q_\rho, U(a), U(b)$, and W^+

$$W = \{(x_1, \dots, x_n) \in \partial U(b) | x_2^2 + \dots + x_n^2 \leq \epsilon_0^2\}$$

And let R_V, R_W be vector fields on V, W :

$$R_V = -x_2 \partial_{x_2} - \dots - x_n \partial_{x_n}$$

$$R_W = x_1 \partial_{x_1}$$

Then, if δ is small enough, for every $-\nabla(f_k - f_l) = -\nabla(\hat{f}_k - \hat{f}_l)$:

$$-\nabla(f_k - f_l)|_V \cdot R_V < 0$$

$$-\nabla(f_k - f_l)|_W \cdot R_W < 0$$

Therefore, for any point $p \in S$, and any choice i, j , we know that $\mathcal{A}_{-(f_i - f_j)}(p) \cap U(b) \subset S$, and in particular the ascending manifold must leave $U(b)$ through W . Therefore, $B_1 \cap U(b), B_2 \cap U(b), \dots, B_m \cap U(b)$ all lie in S , and $B_1 \cap (\partial U(b)), B_2 \cap \partial(U(b)), \dots, B_m \cap \partial(U(b))$ all lie in W . We can separate W into two components: W^+ , whose ascending manifolds lie in $D_{2.49}$, and W^- , whose ascending manifolds are disjoint from $D_{2.49}$.

Now consider the ascending manifold of W^+ for any choice of $-\nabla(f_k - f_l) = -\nabla(\hat{f}_k - \hat{f}_l)$. Define Q'_ρ to be a ρ -neighborhood of the ascending manifold of \bar{b}_j^i for $-\nabla(\bar{f}_k - \bar{f}_l)$, as shown in Figure 6, and define Q_ρ to be the closure of $Q'_\rho - (U(b) \cup U(a))$. Q_ρ then contains no critical points of $\bar{f}_k - \bar{f}_l$ for any k, l , and lies in the descending manifold of \bar{a} for $-\nabla(\bar{f}_k - \bar{f}_l)$. In addition, $W^+ \subset \partial Q_\rho$ if $\epsilon_0 < \rho$.

By Lemma 3.3, if δ is small enough, the ascending manifold of every point in W^+ for $-\nabla(f_k - f_l) = -\nabla(\hat{f}_k - \hat{f}_l)$ will lie within $\frac{1}{2}\epsilon_1$ of the ascending manifold of that point for $-\nabla(\bar{f}_k - \bar{f}_l)$. By Lemma 3.2, if ϵ_0 is small enough, then the ascending manifold of every point of W for $-\nabla(\bar{f}_k - \bar{f}_l)$ will lie within $\frac{1}{2}\epsilon_1$ of the ascending manifold of \bar{b} for $-\nabla(\bar{f}_k - \bar{f}_l)$. Therefore, if δ, ϵ_0 are small enough, the ascending manifold of W^+ for every $-\nabla(f_i - f_j) = -\nabla(\hat{f}_i - \hat{f}_j)$ will lie within Q'_{ϵ_1} . Therefore, if δ, ϵ_0 are small enough, then $B_1 \subset Q'_{\epsilon_1} \cup (M - D_{2.49})$.

Now consider B_2 . Since B_1 consists of the union of the ascending manifolds of W^+ for every choice of height difference functions $-\nabla(f_i - f_j) = -\nabla(\hat{f}_i - \hat{f}_j)$,

then for any choice $-\nabla(f_i - f_j)$, there exists some ascending manifold in B_1 of that function, and every other point in B_1 lies within $2\epsilon_1$ of that ascending manifold. Therefore, by Lemma 3.2, we can choose ϵ_1 to be small enough that B_2 lies within $\frac{1}{2}\epsilon_2$ of B_1 for any choice of ϵ_2 , and therefore $B_2 \subset Q'_{\epsilon_2} \cup (M - D_{2.49})$.

We can then repeat this process for B_3, \dots, B_m , until we obtain that $B_m \subset Q'_{\epsilon_m} \cup (M - D_{2.49})$. What we obtain from this is that, if $\epsilon_m, \epsilon_{m-1}, \dots, \epsilon_1, \epsilon_0$, and δ are all small enough, then B_1, \dots, B_m will be disjoint from D' . Therefore, for the same reasons as for the a chords in Lemma 3.5, there can exist no partial flow tree with a positive special puncture over D' and a negative puncture at a b chord. This concludes the proof.

Lemma 3.7. *If ϵ, ρ are small enough, then there exist no rigid gradient flow trees of Λ whose image passes through D' and which does not have either a switch or a puncture at c .*

Proof: Let Γ be a rigid flow tree in Λ which passes through D' and which has no switch or puncture at c . Let $Y(\Gamma)$ denote the image of Γ in the base space M . We can break Γ at some points $y_1, \dots, y_m \in Y(\Gamma) \cap \partial D'$ into a connected partial flow tree Γ' and a collection of partial flow trees $\Gamma''_1, \dots, \Gamma''_m$, so that Γ' is disjoint from D' , Γ' has negative special punctures at y_1, \dots, y_m , and Γ''_i has a positive special puncture at y_i .

Let r_i, s_i be the sheets of Γ at the point y_i . Generically, every point y_i , if translated to $\hat{\Lambda}$, will be in the ascending manifold of the minimum \hat{d} of r_i, s_i . Therefore, we can find a partial flow tree γ_i in $\hat{\Lambda}$ - *not* in Λ - consisting of a flow line on $-\nabla(f_{r_i} - f_{s_i})$ from y_i to the minimum \hat{d} of sheets r_i, s_i . Since, by construction, the gradient flow $-\nabla(f_{r_i} - f_{s_i})$ is entering D' at y_i , γ_i must cross D' . Observe that Γ' is also a valid partial flow tree in $\hat{\Lambda}$, because it is disjoint from D' , and $\Lambda, \hat{\Lambda}$ agree except over D' . Define $\hat{\Gamma}$ to be the flow tree in $\hat{\Lambda}$ obtained by joining $\Gamma', \gamma_1, \dots, \gamma_m$. By construction, the rigid flow trees of $\hat{\Lambda}$ do not cross D' , so $\hat{\Gamma}$ cannot be rigid.

Recall from section 2.3 that the formal dimension of the moduli space containing a gradient flow tree T is given by:

$$\dim T = (n - 3) + \sum_{i=1}^m (I(p_i) - (n - 1)) - \sum_{j=1}^l (I(q_j) - 1) + \sum_{k=1}^r \mu(v_k) \quad (3.1)$$

Where $n = \dim M$, p_i are the positive punctures, q_j are the negative punctures, v_k are the other vertices, $I(x)$ is the Morse index of the Reeb chord x , and $\mu(z)$ is the Maslov content of a vertex z . Recall further that, if x is a positive special puncture, then $I(x) = n + 1$, and if x is a negative special puncture, then $I(x) = -1$. Finally, recall that, if we divide a flow tree T into two partial flow trees T_1, T_2 , then:

$$\dim T = \dim T_1 + \dim T_2 - (n + 1)$$

Therefore, if we divide T into $m + 1$ partial flow trees T_1, \dots, T_{m+1} , then:

$$\dim T = \dim T_1 + \dots + \dim T_{m+1} - m(n + 1) \quad (3.2)$$

We now apply these formulas to our current case. Each γ_i consists of a positive special puncture, an edge, and a negative puncture at a minimum; therefore:

$$\dim \gamma_i = (n - 3) + ((n + 1) - (n - 1)) - (0 - 1) = n$$

And by equation 3.2:

$$\dim \hat{\Gamma} = \dim \Gamma' + \sum_{i=1}^m \dim \gamma_i - m(n + 1) = \dim \Gamma' - m > 0$$

Therefore:

$$\dim \Gamma' > m \quad (3.3)$$

Now consider $\Gamma''_i, i = 1, \dots, m$. Since Γ''_i does not have a switch or a negative puncture at c , and since by Lemma 3.4 Γ''_i cannot have negative punctures at our a and b Reeb chords, Γ''_i can have no Reeb chords other than minima. Besides minima, it can have Y^0 and Y^1 vertices. Let k be the number of minima in Γ''_i , and let l be the number of Y^1 vertices. Observe that $k \geq l + 1$. Therefore:

$$\begin{aligned} \dim \Gamma''_i &= (n - 3) + ((n + 1) - (n - 1)) - k(0 - 1) + l(-1) = n - 1 + k - l \\ \dim \Gamma''_i &\geq n \end{aligned} \quad (3.4)$$

Therefore, combining equations 3.2, 3.3, and 3.4, we obtain:

$$\dim \Gamma > m + mn - m(n + 1)$$

$$\dim \Gamma > 0$$

Therefore, Γ is not rigid.

Lemma 3.8. *If ϵ, ρ are small enough, then if Γ is a partial flow tree in Λ which has a positive special puncture followed by a switch inside D' and does not have a negative puncture at c , then Γ does not have a Y^1 vertex.*

Proof: The edge emerging from the switch must be on the height difference function $-(f_1 - f_k)$ or $-(f_k - f_0)$. Suppose without loss of generality that it is $-(f_1 - f_k)$. Since there are no cusp edges within D except inside D' , if the partial flow tree has a Y_1 vertex, the height difference function of the edge entering the Y^1 vertex must be on $-(f_i - f_j)$, where $1 \geq i > j \geq k$. But, for a Y^1 vertex, we must have $i > 1 > 0 > j$. This is a contradiction.

Lemma 3.9. *If ϵ, ρ are small enough, then for every rigid gradient flow tree Γ in Λ which has a switch inside D' and does not have a negative puncture at c , there is a unique, second rigid gradient flow tree with the same positive and negative punctures.*

Proof: We can find a point y_0 in Γ immediately above a switch, so that we can break Γ at $y_0 \in D'$ into partial flow trees Γ', Γ'' , where Γ' has a negative special puncture at y_0 , Γ'' has a positive special puncture at y_0 , and Γ'' has only a single switch.

Suppose the switch is from the sheets whose height difference function is $f_0 - f_j$ to the sheets whose height difference function is $f_1 - f_j$ (the following argument works equivalently if it switches from or to other sheets). Let $\Sigma' \subset M$ be the tangency locus of the cusp edge inside D' for the relevant sheets, and let $\mathcal{M}(\Gamma'')$ be the component of the moduli space containing Γ'' . Since Γ'' does not have a negative puncture at c , by Lemmas 3.4 and 3.6 it must consist of a single positive puncture at y_0 , the switch on Σ' , one or more minima at d , and possibly Y^0 vertices. Therefore, for any point $s \in \Sigma'$, we can find a partial flow tree $\Gamma''_s \in \mathcal{M}(\Gamma'')$ that has a switch at s .

Let $P(\Gamma'') \subset M$ denote the set of points $p \in M$ such that there is a partial flow tree in $\mathcal{M}(\Gamma'')$ with a special positive puncture at p . Since we can find a partial flow tree $\Gamma''_s \in \mathcal{M}(\Gamma'')$ for any $s \in \Sigma'$, the ascending manifold of Σ' for the vector field $-\nabla(f_1 - f_j)$ is contained in $P(\Gamma'')$.

Because we have chosen ρ small enough that we can treat f_j as linear over D' , and because $\nabla f_1, \nabla f_0 = 0$ on the cusp edge, Σ' will consist of those points under the cusp edge in D' where $\nabla f_j \in T\Sigma_1$, which will generically be diffeomorphic to S^{n-2} , $n = \dim \Lambda$. Therefore the ascending manifold is diffeomorphic to the cylinder $S^{n-2} \times [0, 1]$; let $P_{\text{SW}}(\Gamma'')$ denote the restriction of $P(\Gamma'')$ to this ascending manifold. Furthermore, since any two points on Σ' are at most 2ρ apart, and ρ is arbitrarily small, by Lemma 3.2 $P_{\text{SW}}(\Gamma'')$ must lie within an arbitrarily small neighborhood of the ascending manifold of $s_0 \in \Sigma'$, where s_0 is the switch in Γ'' . This ascending manifold is a 1-dimensional curve between s_0 and a maximum chord. The upshot is that, since ρ is arbitrarily small, $P_{\text{SW}}(\Gamma'')$ is arbitrarily “narrow”.

Let $\mathcal{M}(\Gamma')$ denote the component of the moduli space containing Γ' , and let $P(\Gamma') \subset M$ denote the set of points $p \in M$ such that there is a partial flow tree in $\mathcal{M}(\Gamma')$ with a special negative puncture at p . If Γ is rigid, then the intersection $P(\Gamma') \cap P_{\text{SW}}(\Gamma'')$ must be zero-dimensional. Since $P_{\text{SW}}(\Gamma'')$ is codimension-1, $P(\Gamma')$ must therefore be codimension- $(n-1)$, that is, 1-dimensional. And, if ρ is small enough, $P_{\text{SW}}(\Gamma'')$ will be narrow enough that the boundary points of $P(\Gamma')$ will not lie in the solid cone whose boundary is $P_{\text{SW}}(\Gamma'') \cup D''$, where D'' is the disk whose boundary is the cusp edge inside D' . Then, $\#(P_{\text{SW}}(\Gamma'') \cap P(\Gamma'))$ must be even. And, for any point $y_1 \in P_{\text{SW}}(\Gamma'') \cap P(\Gamma')$, we can find partial flow trees $\Gamma''_{y_1} \in P_{\text{SW}}(\Gamma''), \Gamma'_{y_1} \in P(\Gamma')$ which have special punctures at y_1 , and connect them together to obtain a rigid gradient flow tree Γ_{y_1} which has the same negative and positive punctures as Γ . This concludes the proof.

Lemma 3.10. *If ϵ, ρ are small enough, then for every rigid gradient flow tree Γ in Λ which has a negative puncture at c , there is a unique, second rigid gradient flow tree with the same positive and negative punctures.*

Proof: The proof works broadly similarly to the proof of Lemma 3.9. Because c is of Morse index 1, Γ cannot have a 2-valent vertex there. Since the ascending manifolds of c of $-\nabla(f_1 - f_0)$ do not intersect any cusp edges, that means that Γ must either be the flow line from a_0^1 to c with no other vertices, or the edge of Γ leading to the vertex at c must begin with a Y^0 vertex, y_0 . If Γ is the flow line, there is a second flow line approaching the other side of c , so assume it is not.

Break Γ at y_0 into $\Gamma_1, \Gamma_2, \Gamma_3$, where Γ_1 has a negative special puncture at y_0 , Γ_2 and Γ_3 have positive special punctures at y_0 , and Γ_3 is the flow line from y_0 to c . Recall that $\mathcal{M}(\Gamma_i)$ denotes the component of the moduli space containing the partial flow tree Γ_i , and that $P(\Gamma_i) \subset M$ denotes the set of points where a partial flow tree in $\mathcal{M}(\Gamma_i)$ has a special puncture. Since Γ is rigid, we know that:

$$\dim(P(\Gamma_1) \cap P(\Gamma_2) \cap P(\Gamma_3)) = 0$$

Further, since c is index-1, its ascending manifold on $-\nabla(f_1 - f_0)$ is a pair of 1-dimensional curves, $S^0 \times (0, 1)$, and these are submanifolds of $P(\Gamma_3)$; call them $P_c(\Gamma_3)$. These are codimension- $(n-1)$, so, generically, $P(\Gamma_1) \cap P(\Gamma_2)$ must be codimension-1.

Consider $P_c(\Gamma_3) - D'$. This consists of a pair of curves, each of which has one boundary point at a_0^1 , and the other boundary point on $\partial D'$; label the boundary points on $\partial D'$ by q_1, q_2 . q_1, q_2 must be within 2ρ of each other. Since $f_i = \hat{f}_i$ outside of D' , the ascending manifolds of q_1, q_2 by $-\nabla(f_1 - f_0)$ equal the ascending manifolds of q_1, q_2 by $-\nabla(\hat{f}_1 - \hat{f}_0)$, and for the function $\hat{f}_1 - \hat{f}_0$, q_1, q_2 lie in the same component of the descending manifold of \hat{a}_0^1 and ascending manifold of \hat{d}_0^1 . Therefore, by Lemma 3.2, for any $\epsilon' > 0$ there exists $\delta' > 0$ such that if $2\rho < \delta'$, then the two components of $P_c(\Gamma_3) - D'$ lie within ϵ' of each other. Therefore, since we can make ρ arbitrarily small, we can ensure that the two curves in $P_c(\Gamma_3)$ are arbitrarily close together outside of D' - and since they must necessarily be within 2ρ of each other inside D' , we can bound the distance between the two curves everywhere.

Therefore, the points in $P(\Gamma_1) \cap P(\Gamma_2) \cap P(\Gamma_3)$ will appear in pairs. For every such point y_1 , we can find partial flow trees $\Gamma'_1 \in \mathcal{M}(\Gamma_1), \Gamma'_2 \in \mathcal{M}(\Gamma_2), \Gamma'_3 \in \mathcal{M}(\Gamma_3)$ that have special punctures at y_1 , and then connect them together with a Y^0 vertex to form a new rigid gradient flow tree Γ' with the same positive and negative punctures as Γ .

Lemma 3.11. *If ϵ, ρ are small enough, then for $x \neq c$, the differential after the Legendrian ambient surgery is given by:*

$$\partial \hat{x} = \widehat{(\partial x)}$$

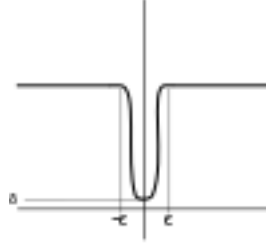


Figure 7: Graph of $\lambda_{\epsilon,\delta}(\mu)$

Proof: Because the only difference between Λ and $\hat{\Lambda}$ is over D' , and by construction all of the rigid flow trees of $\hat{\Lambda}$ avoid D' , all of the rigid flow trees of $\hat{\Lambda}$ are also rigid flow trees of Λ . However, it is possible that there are now new rigid flow trees for Λ that pass through D' . By Lemmas 3.7, 3.9, and 3.10, any such new rigid flow trees appear in pairs, and so their contribution to the differential is canceled out. This concludes the proof.

Proof of Theorem 1.3: This is immediate from Lemma 3.11.

4 Proof of Theorems 1.1, 1.2, and 1.3

4.1 Proof of Theorems 1.1 and 1.3

Recall that, in Theorem 1.1, we defined S to be a hypersurface in M that divides M into R_1 and R_2 , N to be an arbitrarily small neighborhood of S , and $Q_i = R_i \cup N$. We can generically assume that S intersects cusp edges transversely, and that N does not contain any codimension-2 or higher singularities of π_F . Pick a quantity $\delta > 0$ that is less than the action of any existing Reeb chord (new Reeb chords will be created by the pinching isotopy), and choose a tubular neighborhood $S \times [-1, 1]$ of S such that $N = S \times [-1, 1]$, and let μ denote the tubular coordinate.

We define $\lambda_{\epsilon,\delta} : [-\epsilon, \epsilon] \rightarrow \mathbb{R}$ to be a smooth inverted bump function such that:

$$\begin{aligned} 1 &\geq \lambda_{\epsilon,\delta}(\mu) > 0 \\ \lambda_{\epsilon,\delta}(\epsilon) &= \lambda(-\epsilon) = 1 \\ \lambda_{\epsilon,\delta}(0) &= \delta \\ \frac{\partial \lambda_{\epsilon,\delta}}{\partial \mu} &\text{ has the same sign as } \mu \end{aligned}$$

We will ordinarily suppress the subscripts. A graph of this function has the form shown in figure 7.

Then define $l : M \rightarrow \mathbb{R}$ be a smooth function such that:

$$l(x) = \begin{cases} 1 & \text{for } x \notin S \times [-1, 1] \\ \lambda(\mu) & \text{for } x \in S \times [-1, 1] \end{cases}$$

We then define a Legendrian isotopy Λ_t in terms of the front projection $\pi_F(\Lambda)$. Let f_1, \dots, f_m be the sheet functions $M \rightarrow \mathbb{R}$ of the front projection, and define:

$$f_i^t(x) = ((1-t) + tl(x))f_i(x)$$

This defines a Legendrian isotopy Λ'_t , where $\Lambda'_0 = \Lambda$. Although this isotopy captures the primary features we need, we need to make some further small modifications in order to preserve front genericity. Choose a quantity $\epsilon' < \min(\delta, \epsilon/2)$ and perform a C^2 -small perturbation of Λ'_t of order ϵ' in an ϵ' -neighborhood of any cusp edges that intersect N , to make it front generic, as defined in section 2.2. This gives us a new Legendrian isotopy Λ_t . We call this isotopy a **pinching isotopy along S**.

We need to modify our approach to cope with exact Lagrangian cobordisms, since these are not compact and do not have Reeb chords. Recall that we define \hat{L} to be the Legendrian lift of an exact Lagrangian cobordism L ; that $\hat{S} \subset M \times \mathbb{R}^+$ is a hypersurface such that each time-slice is a hypersurface in M and $\pi_M^{-1}(\hat{S})$ is disjoint from Σ_1 ; and that \hat{N} is an arbitrarily small neighborhood of \hat{S} . A Legendrian isotopy of \hat{L} descends to an exact Lagrangian *homotopy* of L . This is an exact Lagrangian isotopy if and only if there are no Reeb chords at any time in the Legendrian isotopy. This is the reason for the restriction that \hat{S} cannot cross any cusp edges or self-intersections of $\pi_F(\hat{L})$ in the statement of Theorem 1.3; we claim that this is a sufficient (though not necessary) condition to allow us to find a pinching isotopy of \hat{L} that descends to an exact Lagrangian isotopy.

Instead of $\lambda(\mu)$, we use $\lambda(\mu, \tau)$, where τ is the cylindrical coordinate. Let T_-, T_+ be the cylindrical coordinates such that $L \cap (J^1(M) \times (0, T_-))$, $L \cap (J^1(M) \times (T_+, \infty))$ are the cones over Λ_-, Λ_+ . Let $\delta > 0$ be some quantity smaller than the action of the smallest Reeb chord of $\Lambda_- = \partial\mathcal{E}_-(L)$. We require that $\lambda(\mu, \tau)$ obey the same requirements as $\lambda(\mu)$, and, in addition:

$$\begin{aligned} \frac{\partial \lambda}{\partial \tau}(\mu, \tau) &\geq 0 \\ \lambda(\mu, \tau) &= \frac{t}{T_-} \lambda(\mu, T_-) \text{ for } \tau \leq T_- \\ \lambda(\mu, \tau) &= \frac{t}{T_+} \lambda(\mu, T_+) \text{ for } \tau \geq T_+ \end{aligned}$$

We then define an isotopy on \hat{L} in a precisely analogous fashion as we did for Λ .

Lemma 4.1. *For any exact Lagrangian cobordism L , given a hypersurface $\hat{S} \subset M \times \mathbb{R}^+$ such that each time-slice is a hypersurface in M , and there is a neighborhood \hat{N} of \hat{S} such that no cusp edges lie above \hat{N} , the isotopy induced by \hat{L} descends to an exact Lagrangian isotopy.*

Proof: Let f_i, f_j be sheet functions of $\pi_F(\hat{L})$ prior to the pinching isotopy, $f_i > f_j$. Since, by definition, \hat{N} may not cross a cusp edge, we further know that:

$$\left. \frac{\partial}{\partial \tau} (f_i - f_j) \right|_{\hat{N}} > 0$$

During the pinching, within $\hat{S} \times [-\epsilon, \epsilon]$ f_i, f_j are replaced by $((1-t) + t\lambda(\mu, \tau))f_i(x, t), ((1-t) + t\lambda(\mu, \tau))f_j(x, t)$. Then, within \hat{N} , observe that:

$$\frac{\partial}{\partial \tau} (((1-t) + t\lambda(x, t))(f_i - f_j)) = t \frac{\partial \lambda}{\partial \tau} (f_i - f_j) + (((1-t) + t\lambda(x, t)) \frac{\partial}{\partial t} (f_i - f_j))$$

By construction, all of the terms of on the right hand side of the equation are positive, so there are no Reeb chords within \hat{N} at any value of t .

Once we have performed the pinching isotopy, we can then prove the main theorem, outsourcing the detailed analysis to appendix A:

Proof of Theorem 1.1 and 1.3: Define $u_\sigma = (q_\sigma, p_\sigma)$, where q_σ is the base space coordinate and p_σ is the cofiber coordinate. As discussed in Appendix A, after adding some additional punctures to the domain Δ_m of u_σ to obtain Δ_r , we can divide Δ_r of u_σ into overlapping sets $D_0(\sigma) \cup D_1(\sigma) \cup D_2(\sigma)$, where $D_0(\sigma)$ maps to a neighborhood away from the singularities of π_F , $D_1(\sigma)$ maps to a neighborhood of the cusp edges, and $D_2(\sigma)$ maps to a neighborhood of the higher-codimension singularities of π_F . We can further find $W_0(\sigma) \subset D_0(\sigma), W_1(\sigma) \subset D_1(\sigma)$. Then according to lemma A.13, $u_\sigma(W_0(\sigma))$ converges to a collection of discrete points as $\sigma \rightarrow 0$, and according to Lemma A.14, over $D_0(\sigma) - W_0(\sigma)$, as $\sigma \rightarrow 0$:

$$\nabla_t p_\sigma \rightarrow (\sigma b_1(q_\sigma) - \sigma b_0(q_\sigma))$$

$$\nabla_\tau p_\sigma \rightarrow 0$$

Since u_σ is pseudoholomorphic, this in turn gives us:

$$\nabla_\tau q_\sigma \rightarrow \sigma b_1(q_\sigma) - \sigma b_0(q_\sigma)$$

$$\nabla_t q_\sigma \rightarrow 0$$

Where b_i are the gradients of sheet height functions. This tells us that $u_\sigma(\partial \Delta_r)$ converges to a gradient flow over $D_0(\sigma) - W_0(\sigma)$.

Similarly, we define u_σ^T to be the projection of u_σ restricted to $D_1(\sigma)$ to $J^1(\pi(\Sigma_1))$, where $\Sigma_1 \subset \Lambda$ is the set of cusp edges. According to lemmas A.15

and A.16 this converges to a gradient flow in the projection of Λ_σ to $J^1(\pi(\Sigma_1))$. A secondary consequence of this is also that $b_1 \neq b_0$.

Now consider q_σ as $\sigma \rightarrow 0$. Outside of an arbitrarily small neighborhood of the codimension-2 subset $\pi_{\mathbb{C}}(\Sigma_2) \subset T^*M$, $q_\sigma(\partial\Delta_r)$ converges to a collection of gradient flow lines. Since the neighborhood of $\pi_{\mathbb{C}}(\Sigma_2)$ is arbitrarily small, we can assume that it is disjoint from any points in $\pi_{\mathbb{C}}(\Lambda_\sigma)$ which correspond to a crossing of sheets in the front projection $\pi_F(\Lambda_\sigma)$. Recall from section 2.2 that $u_\sigma(\partial\Delta_r)$ will have a continuous lift to $\Lambda \subset J^1(M)$. As a consequence, if Δ_r has only a single positive puncture, $q_\sigma(\partial\Delta_r)$ may not converge to a flow line along a pair of sheets whose front projections cross each other, since this would imply the existence of a second positive puncture.

Now suppose we have pinched Λ along S to form Λ' using arbitrarily small δ, ϵ , and suppose we have a pseudoholomorphic disk u_σ such that u_σ has punctures over both $M - Q_1$ and $M - Q_2$. By construction, ∂N will be disconnected; let $\partial_1 N, \partial_2 N$ be two components of ∂N . This implies that, as $\sigma \rightarrow 0$, $q_\sigma(\partial\Delta_r) \cap N$ converges to the embedding of a tree whose edges follow the gradient flow of height difference functions and which intersects both $\partial_1 N$ and $\partial_2 N$. (Note that, although this is a tree that follows the gradient flows, it is technically not necessarily a *gradient flow tree*, in the sense of [6].) We can therefore truncate the tree to find a piecewise smooth map $\gamma : [0, 1] \rightarrow N$ such that $\gamma(0) \in \partial N_1, \gamma(1) \in \partial N_2$, and except at finitely many points $v_1, \dots, v_m, \gamma' = -\nabla(f_i - f_j)$ for some height functions f_i, f_j , and $f_i > f_j$. Let $h : ([0, 1] - \{v_1, \dots, v_m\}) \rightarrow \mathbb{R}^+$ be the function that maps $t \in [0, 1]$ to $f_i(\gamma(t)) - f_j(\gamma(t))$, where f_i, f_j are the corresponding height functions at $\gamma(t)$. Because $\gamma'(t) = -\nabla(f_i - f_j)$, $h'(t) < 0$. Furthermore, by construction, $h(v_k - \epsilon) > h(v_k + \epsilon)$ for $\epsilon > 0$. Therefore, h is monotone decreasing.

4.2 Proof of Theorem 1.2

The argument in this section is taken essentially from [9], Sec. 3.

Proof of Theorem 1.2: Λ is a Legendrian submanifold pinched as in theorem 1.1 along a neighborhood N of a hypersurface S dividing M into Q_1, Q_2 , and that ϵ is an augmentation of $\mathcal{A}(\Lambda)$.

Define i_1, i_2, j_1, j_2 to be the inclusion maps:

$$\begin{array}{ccc} \mathcal{A}(\Lambda) & \xleftarrow{i_1} & \mathcal{A}(\Lambda)|_{Q_1} \\ i_2 \uparrow & & \uparrow j_1 \\ \mathcal{A}(\Lambda)|_{Q_2} & \xleftarrow{j_2} & \mathcal{A}(\Lambda)|_N \end{array}$$

Observe that $i_1 \circ j_1 = i_2 \circ j_2$. Define $\epsilon_{Q_1} = \epsilon \circ i_1, \epsilon_{Q_2} = \epsilon \circ i_2, \epsilon_N = \epsilon \circ i_1 \circ j_1 = \epsilon \circ i_2 \circ j_2$. These are obviously augmentations, since $\epsilon_X \circ \partial|_X = (\epsilon \circ \partial)|_X = 0$.

Now, consider the maps $j_1 \oplus j_2$ and $i_1 + i_2$. Observe that:

$$(i_1 + i_2) \circ (j_1 \oplus j_2) = (i_1 \circ j_1) + (i_2 \circ j_2) = 0$$

Since we are working over \mathbb{Z}_2 . Therefore, the image of $j_1 \oplus j_2$ lies in the kernel of $i_1 + i_2$. Now observe that, if (x, y) is in the kernel of $i_1 + i_2$, that implies that $x = y$, meaning that x is in the intersection of $\mathcal{A}(\Lambda)|_{Q_1}$ and $\mathcal{A}(\Lambda)|_{Q_2}$, which is $\mathcal{A}(\Lambda)|_N$. Therefore the kernel of $i_1 + i_2$ equals the image of $j_1 \oplus j_2$ and we have a long exact sequence.

Just as we have a Seifert-van Kampen theorem for both Legendrian submanifolds and exact Lagrangian cobordisms, there is an extension of the Mayer-Veitoris theorem to exact Lagrangian cobordisms. Let L be an exact Lagrangian cobordism from Λ_+ to Λ_- , and let Φ_L be the cobordism map. If Λ_- has an augmentation ϵ , this induces an augmentation $\epsilon \circ \Phi_L$ on Λ_+ . Then, after splashing, theorem 1.4 holds for the linearized chain complexes of Λ_+, Λ_- as well as the differential graded algebra.

A Convergence of Disk Boundaries to Flow Lines

This section is derived, with some modifications and a great many omissions, from [6]. Sections 3.1 and 3.2 are derived entirely from [Ek, Sec. 4.2]. Section 3.3 is derived from [6], Sec. 5.1, but we make the reasoning more explicit. The remaining sections are derived from [6], Sec. 5.2 and 5.3, but with modifications to allow higher-dimensional singularities in the front projection. For simplicity's sake, we will provide the original source for each lemma in parentheses next to the lemma.

Let $s_\sigma : J^1(M) \rightarrow J^1(M)$, $s_\sigma(x, y, z) = (x, \sigma y, \sigma z)$ be the scaling of the cofiber and z components by $\sigma > 0$. We define:

$$\tilde{\Lambda}_\sigma = s_\sigma(\Lambda)$$

We begin by defining some arbitrarily small modifications to $\tilde{\Lambda}_\sigma$ in sections A.1 and A.2. These modifications will produce Λ_σ , which is what we will work with after those sections. Λ_σ allows us to obtain a modified version of the monotonicity lemma in section A.3, which will still hold even as σ varies. In section A.4, we define the slit model of the disk. In section A.5, we use the slit model and the monotonicity lemma to bound the derivatives of the map of the pseudoholomorphic disk in terms of σ . In section A.6, we add punctures to the boundary of the disk, which allows us to restrict our attention to the disk away from an arbitrarily small neighborhood of the codimension-2 singularities of π_F . In section A.7, we show that, away from this neighborhood, the boundary of the disk converges to a flow line of a height difference function as $\sigma \rightarrow 0$. In section 4.1, we use this result to prove theorems 1.1 and 1.3. In section 4.2, we extend this result to linearized contact homology, proving theorem 1.2

A.1 Deformations of the Legendrian - The Metric

Recall that section 2.2 defines $\Sigma_k \subset \Lambda$ as the codimension- k components of the singularities of the front projection π_F . Define $U(k, d)$ to be a product

neighborhood of Σ_k of radius d inside Λ , and define $N(k, d)$ to be a product neighborhood of Σ_k of radius d in T^*M .

Let \tilde{g} be a metric on M such that the self-intersections of $\pi_M(\Sigma_k)$ are orthogonal. Let $\epsilon_1 > 0$ be a deformation parameter. Define b to be the restriction of \tilde{g} to $\pi_M(\Sigma_1)$. Using b and \tilde{g} we can produce a metric g on $U(1, \epsilon_1)$ which can be made arbitrarily close to \tilde{g} by choosing ϵ_1 small enough, and such that if $m \in \Sigma_1 - \Sigma_2$ then there exists a coordinate patch around m disjoint from Σ_2 with coordinates $(q, s) \in \mathbb{R}^{n-1} \times \mathbb{R}$, such that:

$$\begin{aligned} \Sigma_1 \text{ corresponds to } \{s = 0\} \\ g_{q,s} = \sum_{i,j} (b_{i,j})(q) dq_i \otimes dq_j + ds \otimes ds \end{aligned}$$

Now that we have established this product metric and coordinates, we move on to cusp rounding inside $U(1, \epsilon_1)$.

A.2 Deformations of the Legendrian - Cusp Rounding

Let $m \in \pi_M(\Sigma_1)$, and pick a coordinate patch $(q, s) \in \mathbb{R}^{n-1} \times \mathbb{R}$ around m . We have two kinds of sheets over m : those on which π_M is an immersion, and those with a cusp edge singularity. We call the former an **unfolded sheet**, and the latter a **folded sheet**.

From section A.1, we have local coordinates (q, s, κ, ν, z) of $J^1(M)$, where $(q, s) \in \mathbb{R}^{n-1} \times \mathbb{R}$, κ_i is the cofiber coordinate of q_i , ν is the cofiber coordinate of s , and z is the \mathbb{R} coordinate. Then an unfolded sheet can be parameterized locally as the graph of a function f :

$$(q, s) \rightarrow (q, s, \sigma \partial_q f, \sigma \partial_s f, \sigma f(q, s))$$

We perform a small Legendrian isotopy for $|s| < \epsilon_1$, replacing f with its Taylor polynomial of degree 1 in s :

$$\begin{aligned} f(q, s) &= a(q) + sh(q) + \dots \\ (q, s) &\rightarrow (q, s, \sigma \partial_q a + \sigma s \partial_q h, \sigma h, \sigma(a + sh)) \end{aligned} \tag{A.1}$$

This is a Legendrian isotopy that introduces no new Reeb chords.

Next we perform a similar isotopy for folded sheets. We assume without loss of generality that the projection of the folded sheet lies in $\{s \geq 0\}$. Then the sheet can be parameterized locally as the graph of a function f :

$$(q, s) \rightarrow \left(q, \frac{1}{2}s^2, \sigma \partial_q f, \sigma \partial_s f, \sigma f \right)$$

Where:

$$\begin{aligned} \frac{\partial f}{\partial s}(q, 0) &= 0 \\ \frac{\partial^3 f}{\partial s^3}(q, 0) &\neq 0 \end{aligned}$$

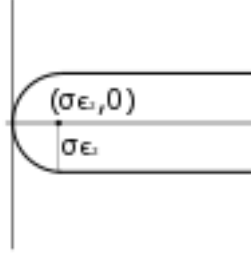


Figure 8: Graph of c_σ in $T^*\mathbb{R}$

We perform an isotopy over $|s| \leq \epsilon_1$ to replace f with its Taylor polynomial of degree 3 in s , obtaining:

$$(q, s) \rightarrow \left(q, \frac{1}{2}s^2, \sigma \left(\partial_q a + \frac{1}{2}s^2 \partial_q \beta + \frac{1}{3}s^3 \partial_q \alpha \right), \right. \\ \left. \sigma(\beta + s\alpha), \sigma \left(a + \frac{1}{2}s^2 \beta + \frac{1}{3}s^3 \alpha \right) \right) \quad (\text{A.2})$$

Where α, β, a depend only on q , and $\alpha(q) \neq 0$.

We follow this with an additional isotopy inside an even smaller neighborhood of the cusp edge. For notational simplicity, we begin with the case $\dim \Lambda = 1$, where the cusp edge has the form:

$$s \rightarrow \left(\frac{1}{2}s^2, \sigma s, \frac{1}{3}\sigma s^3 \right)$$

Where $-\delta \leq s \leq \delta$. Fix $\epsilon_2 \ll \delta$. We define Θ_σ to be the projection of the above cusp edge to $T^*\mathbb{R}$ for $0 \leq x \leq l$, where x is the base space coordinate and y is the cofiber coordinate, and where $l > \sigma\epsilon_2$. We define c_σ to be the curve in $T^*\mathbb{R}$ given by a half-circle of radius $\sigma\epsilon_2$ centered at $(\sigma\epsilon_2, 0)$, connected to a pair of horizontal line segments $y = \pm\sigma\epsilon_2$ from $x = \sigma\epsilon_2$ to $x = l$. Explicitly, this has the formula:

$$s \rightarrow \left(\begin{array}{ll} \left(\frac{1}{2}s^2, -\sigma\epsilon_2 \right) & s \in (-\infty, -\sqrt{2\sigma\epsilon_2}) \\ \left(\frac{1}{2}s^2, \frac{1}{2}s\sqrt{4\sigma\epsilon_2 - s^2} \right) & s \in (-\sqrt{2\sigma\epsilon_2}, \sqrt{2\sigma\epsilon_2}) \\ \left(\frac{1}{2}s^2, \sigma\epsilon_2 \right) & s \in (\sqrt{2\sigma\epsilon_2}, \infty) \end{array} \right) \quad (\text{A.3})$$

Restricted to $0 \leq x \leq l$.

The area bounded by the curve c_σ is $\frac{1}{2}\pi\epsilon_2^2\sigma^2 + 2l\sigma\epsilon_2$. The area bounded by the curve Θ_σ is $\frac{4\sqrt{2}}{3}\sigma l^{3/2}$. The two curves intersect each other at $x = \epsilon_2^2/2$. Therefore, we can find a Hamiltonian isotopy supported in $-\epsilon_2^2 \leq x \leq 10\epsilon_2^2$ which deforms Θ_σ into a smoothed version of c_σ for $0 \leq x \leq \frac{1}{2}\epsilon_2^2$ and so that the first and second derivatives of the new curve are bounded by $K\sigma$ for some

$K > 0$. Therefore, we can lift this isotopy to a Legendrian isotopy, giving us a new curve \tilde{c}_σ parameterized by:

$$s \rightarrow \left(\frac{1}{2}s^2, \gamma_\sigma(s), \Psi_\sigma(s) \right) \quad (\text{A.4})$$

Where $\gamma_\sigma, \Psi_\sigma$ have the properties:

$$\begin{aligned} \gamma_\sigma(0) &= \Psi_\sigma(0) = 0 \\ \gamma_\sigma(s) &= \sigma s \text{ for } |s| \geq 100\epsilon_2^2 \\ \Psi_\sigma(s) &= \frac{1}{3}\sigma s^3 \text{ for } |s| \geq 100\epsilon_2^2 \end{aligned}$$

We can extend this Legendrian isotopy naturally to $\dim \Lambda > 1$, giving us a Legendrian isotopy restricted to a very small neighborhood of our cusp edge which changes our folded sheets from equation A.2 to the form:

$$\begin{aligned} (q, s) &\rightarrow \left(q, \frac{1}{2}s^2, \sigma \left(\partial_q a + \frac{1}{2}s^2 \partial_q \beta \right) + \Psi_\sigma(s) \partial_q \alpha, \right. \\ &\quad \left. \sigma \beta + \alpha \gamma_\sigma(s), \sigma \left(a + \frac{1}{2}s^2 \beta \right) + \Psi_\gamma(s) \alpha \right) \end{aligned} \quad (\text{A.5})$$

Formally, we denote this deformed Legendrian by $\Lambda_\sigma(\epsilon_0, \epsilon_1, \epsilon_2)$. We will ordinarily omit the deformation parameters, and write it simply as Λ_σ .

To sum up: our deformed Legendrian submanifold Λ_σ will be equal to the cofiber-scaling of Λ away from the cusp edges. Near the cusp edges, sheets that are not folded in the cusp edge will be approximated linearly, while sheets that are folded will be approximated by their degree-3 Taylor polynomial. Finally, in an even smaller neighborhood inside of that neighborhood, the cusp edges will be replaced by semicircles of radius $\mathcal{O}(\sigma)$ in the Lagrangian projection.

A.3 The Monotonicity Lemma

The purpose of these deformations is to ensure that we have a useable version of the monotonicity lemma. This section is based on the treatment in [6], Sec. 5.1, though made more explicit, and in [1], Ch. 5, Sec. 4, though [1] considers only fixed Lagrangians.

Let $p \in T^*M$, and let $B(p, r)$ be an r -ball around p . The standard monotonicity lemma says that if we have a J -holomorphic map $u : (D, \partial D) \rightarrow (B(p, r), \partial B \cup \pi_{\mathbb{C}}(\Lambda_\sigma))$, then there exists a constant $C' > 0$ such that:

$$\text{Area}(u(D)) \geq C' r^2$$

The problem is that C' depends on $\pi_{\mathbb{C}}(\Lambda_\sigma)$, which in turn depends on σ .

Even with our deformations, we cannot retrieve the full version of the monotonicity lemma. What we *can* do, however, is get a version that is good enough. Recall that by the definition of a tame almost complex symplectic manifold,

there exists constants r_0 and C_1 such that if γ is a loop in T^*M contained in a ball $B(x, r)$, $r \leq r_0$, then γ bounds a disc in $B(x, r)$ of area less than $C_1 \text{length}(\gamma)^2$. Then:

Lemma A.1. *There exists a constant C_2 such that, for any Riemannian surface with boundary D and any pseudoholomorphic map $f : D \rightarrow T^*M$, if $f(D) \subset B(p, r_0)$ then:*

$$\text{area}(f(D)) \leq C_2 (\text{length}(f(\partial D)))^2 \quad (\text{A.6})$$

In addition, there exists a constant C_3 such that, if $f(\partial D) \subset \partial B(p, r_0)$, and $p \in f(D)$, then:

$$\text{area}(f(D)) \geq C_3 r^2 \quad (\text{A.7})$$

Proof: See [1], Ch. 5, Prop. 4.3.1.i and ii. In particular, the second statement is the general form of the monotonicity lemma for symplectic manifolds.

Note that C_1, C_2, C_3 do not depend on σ , or even on Λ . They are properties of the symplectic manifold T^*M and its associated symplectic form, almost complex structure, and Riemannian metric.

Lemma A.2. *If ϵ_2 is small enough, then there exists a constant C_4 independent of σ such that, if D is a disk in \mathbb{C} and $u : (D, \partial D) \rightarrow (B(x, r), \partial B \cup \pi_{\mathbb{C}}(\Lambda_{\sigma}))$ is a pseudoholomorphic map, $r \leq \sigma r_0$, where r_0 is the r_0 defined in remark 5.1, then:*

$$\text{Area}(f(D)) \leq C_4 r^2 \quad (\text{A.8})$$

Proof: In what follows, we will refer to the “cusp edges” and “sheets” of the Legendrian, even though we are working with the Lagrangian projection $\pi_{\mathbb{C}}(\Lambda)$. These both refer to the projection of the corresponding objects in the front projection to the Lagrangian projection.

Choose $\delta \leq r_0$ small enough that no solid ball in T^*M of radius δ intersects more than one sheet of $\pi_{\mathbb{C}}(\Lambda_1)$ unless it contains a cusp edge or a double point, in which case it intersects at most two sheets. In addition, choose δ small enough that we can always find a coordinate chart of T^*M containing any solid ball of radius δ . The subscript Λ_1 is used to indicate $\sigma = 1$. If ϵ_2 is small enough, and $\delta < \epsilon_2$, then any ball $B(p, \sigma\delta)$, $p \in \pi_{\mathbb{C}}(\Lambda_{\sigma})$, will intersect $\pi_{\mathbb{C}}(\Lambda_{\sigma})$ in at most one sheet unless it contains a cusp edge or a double point, in which case it will intersect at most two sheets, thanks to the cusp rounding conducted in section A.2.

We can regard $\pi_{\mathbb{C}}(\Lambda_{\sigma})$ as the union of the graphs of a collection of sheet functions:

$$f_i : \pi_M(U_i) \rightarrow \mathbb{R}^n$$

Where $U_i \subset \Lambda_{\sigma}$. Because Λ_{σ} is compact, outside of the $\sigma\epsilon_2$ -neighborhood of the cusp edges, we can bound $|\nabla f_i| \leq C'\sigma$ for some constant C' . In addition, inside

the $\sigma\epsilon_2$ -neighborhood of $\pi_{\mathbb{C}}(\Sigma_1)$, but outside of a $\sigma\epsilon_2$ -neighborhood of $\pi_{\mathbb{C}}(\Sigma_2)$, we can instead describe $\pi_{\mathbb{C}}(\Lambda_\sigma)$ as the graph of a function $g_i(v_1, u_2, \dots, u_n)$, and bound $|\nabla g_i| \leq C''$.

Let $\rho_1, \rho_2 \in \partial B(p, r) \cap \pi_{\mathbb{C}}(\Lambda_\sigma)$, $r \leq \sigma\delta$. We begin with the case where $B(p, r)$ does not intersect a double point or the $\sigma\epsilon_2$ -neighborhood of a cusp edge. Let $f_i : \pi_M(B) \rightarrow \mathbb{R}^n$, where $f_i = (f_i^1, \dots, f_i^n)$, be the sheet function. Pick a coordinate chart in M around $\pi_M(p)$, with $\pi_M(\rho_i) = (\rho_i^1, \dots, \rho_i^n)$. Then we can define a path $\gamma : [0, 1] \rightarrow \pi_{\mathbb{C}}(\Lambda_\sigma)$ that connects ρ_1 to ρ_2 by:

$$\begin{aligned}\gamma(t) &= (t\rho_1^1 + (1-t)\rho_2^1, \dots, t\rho_1^n + (1-t)\rho_2^n, f_i(t\pi_M(\rho_1) + (1-t)\pi_M(\rho_2))) \\ \gamma'(t) &= (\pi_M(\rho_1) - \pi_M(\rho_2), \nabla f_i^1 \cdot (\pi_M(\rho_1) - \pi_M(\rho_2)), \dots, \nabla f_i^n \cdot (\pi_M(\rho_1) - \pi_M(\rho_2))) \\ |\gamma'(t)|^2 &\leq (1 + (C'\sigma)^2)(2r)^2\end{aligned}$$

Therefore, $\text{length}(\gamma) \leq 2r\sqrt{1 + (C'\sigma)^2}$.

A similar argument allows us to bound the length of a path by $4r\sqrt{1 + (C'\sigma)^2}$ if the ball intersects a double point.

If the ball intersects the $\sigma\epsilon_2$ -neighborhood of the cusp edge, then pick a coordinate chart in T^*M corresponding to the product neighborhood of the cusp edge, with $\rho_i = (\rho_i^1, \dots, \rho_i^n, \nu_i^1, \dots, \nu_i^n)$, and let g be the cusp edge function. We define π' to be the projection from $(u, v) \rightarrow (u_2, \dots, u_n, v_1)$. We can define a path $\gamma : [0, 1] \rightarrow \pi_{\mathbb{C}}(\Lambda_\sigma)$ that connects ρ_2 to ρ_1 by:

$$\begin{aligned}\gamma(t) &= (g_1(t\pi'(\rho_1) + (1-t)\pi'(\rho_2)), t\rho_1^2 + (1-t)\rho_2^2, \dots, t\rho_1^n + (1-t)\rho_2^n, \\ &\quad t\nu_1^1 + (1-t)\nu_2^1, g_2(t\pi'(\rho_1) + (1-t)\pi'(\rho_2)), \dots) \\ \gamma'(t) &= (\nabla g_1 \cdot (\pi'(q_1) - \pi'(q_2)), q_1^2 - q_2^2, \dots, q_1^n - q_2^n, r_1^1 - r_2^1, \nabla g_2 \cdot (\pi'(q_1) - \pi'(q_2)), \dots) \\ |\gamma'(t)|^2 &\leq (1 + (C'')^2)(2r)^2\end{aligned}$$

Therefore, $\text{length}(\gamma) \leq 2r\sqrt{1 + (C'')^2}$. C'' is independent of σ because, as a result of the cusp rounding and the fact we are working in an arbitrarily small neighborhood of the cusp edge, g is independent of σ . Therefore, we can find C such that, for any ball of radius $r \leq \delta\sigma$, any any two points $\rho_1, \rho_2 \in B(p, r) \cap \pi_{\mathbb{C}}(\Lambda_\sigma)$, there exists a path in $\pi_{\mathbb{C}}(\Lambda_\sigma)$ linking ρ_1 to ρ_2 of length less then or equal to Cr .

Now let $B(p, r)$, $r \leq \delta\sigma$ be an arbitrary ball intersecting Λ_σ , and let u be a pseudoholomorphic map of an open disk $(D, \partial D) \rightarrow (B(p, r), \partial B \cup \pi_{\mathbb{C}}(\Lambda_\sigma))$. Then $\partial(u(D)) - \pi_{\mathbb{C}}(\Lambda_\sigma)$ is a union of arcs α_j in $\partial B(p, r)$ from $\pi_{\mathbb{C}}(\Lambda_\sigma)$ to $\pi_{\mathbb{C}}(\Lambda_\sigma)$, which have length less then or equal to πr . Let α'_j be arcs in $\pi_{\mathbb{C}}(\Lambda_\sigma) \cap B(p, r)$ closing α_j . Then they have length less then or equal to Cr . Therefore, by lemma A.1, $\alpha_j \cup \alpha'_j$ bound a union of disks W of area less then or equal to $C_2 r^2$, where C_2 is a constant that depends only on T^*M . Therefore, $W \cup u(D)$ is a surface with boundary on $\partial B(p, r) \cap \pi_{\mathbb{C}}(\Lambda_\sigma)$.

$B(p, r) \cap \pi_{\mathbb{C}}(\Lambda_\sigma)$ must be a disk or pair of disks by our restrictions on r , so it is contractible. By the fact that $B(p, r) \cap \pi_{\mathbb{C}}(\Lambda_\Sigma)$ is contractible and the fact that $\|w\| \leq 1$, we obtain that the area of $u(D)$ is less then or equal to $C_4 r^2$, where C_4 is another constant that depends only on T^*M .

Lemma A.3 (Monotonicity Lemma). *If ϵ_2 is small enough, then there exists constants $C, \delta > 0$ which are independent of σ such that, if $0 < r \leq \sigma\delta$, then for any non-constant J -holomorphic map $u : (D, \partial D) \rightarrow (B(p, r), \partial B \cup \pi_{\mathbb{C}}(\Lambda_{\sigma}))$, such that p is in the image of u , and where D is an open surface, then:*

$$\text{Area}(u(D)) \geq Cr^2 \quad (\text{A.9})$$

Proof: Define $S_t = u^{-1}(B(x, t))$ for $t \leq r$, $a(t) = \text{area}(u|_{S_t})$. Since u is smooth, Sard's theorem ([15], Theorem II.3.1) implies that S_t is a sub-surface with C^1 boundary $\partial S_t = u^{-1}(\partial(B(x, t)))$ for almost all t . Define $l(t) = \text{length}(u(\partial S_t))$. $a(t)$ is an absolutely continuous function, and $a'(t) = l(t)$ for almost all t . Then by lemma A.2, we have $a(t) \leq C_4(l(t))^2$ for almost all $t \leq r$. Since u is not constant and the image of u contains p , $a(t) > 0$ for $t > 0$. Thus, for $t > 0$ we know:

$$\frac{d}{dt} \left(\sqrt{a(t)} \right) = \frac{a'(t)}{2\sqrt{a(t)}} \geq \frac{l(t)}{2\sqrt{C_4(l(t))^2}} = \frac{1}{2\sqrt{C_4}}$$

Therefore, integrating the above from $t = 0$ to r , we obtain:

$$\text{Area}(u(D)) = a(r) \geq \frac{r^2}{4C_4}$$

Which implies equation A.9.

A.4 The Slit Model of the Disk

Recall that D_m is the unit disk in \mathbb{C} with m -punctured boundary. We define Δ_m to be the subset of $\mathbb{R} \times [0, m-1]$ given by deleting $m-2$ horizontal slits of width $\epsilon < 1$, where each slit ends in a half-circle. Δ_m has a canonical complex and symplectic structure inherited from $\mathbb{R}^2 = \mathbb{C}$.

Lemma A.4. ([6], Lemma 2.2) Δ_m is biholomorphic to D_m .

Proof: See [6], Lemma 2.2.

We will work in the slit model of the disk for most of what follows.

A.5 Lemmas Bounding the Derivative

Recall from the introduction to this section that $s_{\sigma}(x, y, z) = (x, \sigma y, \sigma z)$ is the scaling of the cofiber and z components by $\sigma > 0$. Let $J_{\sigma} = (s_{\sigma})_*^{-1} \circ J \circ (s_{\sigma})^*$. Then, there is a bijection between J -holomorphic disks on $\pi_{\mathbb{C}}(\Lambda)$ and J_{σ} -holomorphic disks on $\pi_{\mathbb{C}}(\Lambda_{\sigma})$.

Let $u : (\Delta_m, \partial\Delta_m) \rightarrow (T^*M, \pi_{\mathbb{C}}(\Lambda))$ be a pseudoholomorphic map. Let u_σ be the corresponding map $(\Delta_m, \partial\Delta_m) \rightarrow (T^*M, \Lambda_\sigma)$, which can be parameterized by $u_\sigma = (p_\sigma, q_\sigma)$, where p_σ is the map to the base space and q_σ is the map to the cofiber.

Recall from section 2.2 that $\Sigma_k \subset \Lambda$ denotes the codimension- k component of the singularities of the front projection π_F , that $U(k, d) \subset \Lambda$ is the product neighborhood of radius d around Σ_k in Λ , and that $N(k, d) \subset T^*M$ is the product neighborhood of radius d around Σ_k in T^*M . We analogously have $(\Sigma_k)_\sigma, (U(k, d))_\sigma \subset \Lambda_\sigma, (N(k, d))_\sigma \subset T^*M$; however, in what follows we will omit the subscript. There is a projection map $\pi_\Sigma : T^*U(1, d) \rightarrow T^*\Sigma_1$. Define $u^T = \pi_\Sigma \circ u, u_\sigma^T = \pi_\Sigma \circ u_\sigma$, with u, u_σ restricted to the pre-images of $T^*U(1, d)$. Define $V(k, d) = U(k, d) - U(k + 1, \epsilon_0)$, where $\epsilon_0 > 0$ is an extremely small deformation parameter.

Throughout all that follows, we will restrict u_σ to vertical lines contained in the pre-image of $T^*M - T^*N(2, \epsilon_3)$, where ϵ_3 is arbitrarily small. We denote the restricted domain of vertical lines by D_σ .

Next, we put bounds on $|Du_\sigma|$. To do this, we will use O -notation, where $f(\sigma) = O(\sigma)$ means that:

$$\lim_{\sigma \rightarrow 0} \frac{f(\sigma)}{\sigma} \text{ is finite}$$

Lemma A.5. ([6], Lemma 5.2 and 5.4) *For any J -holomorphic map $u : (\Delta_m, \partial\Delta_m) \rightarrow (T^*M, \Lambda)$, there exist constants $C', C'' > 0$ independent of σ such that the symplectic area of $u_\sigma(D_\sigma)$ is less than $C'\sigma$ and the length of $u_\sigma(D_\sigma \cap \partial\Delta_m)$ is less than C'' for any σ .*

Proof: By Stokes' Theorem:

$$\int_{u(D_\sigma)} \omega = \int_{\partial u(D_\sigma)} \beta$$

Since β is the tautological 1-form, and $|q_\sigma|$ is scaled by σ , this integral equals $C'\sigma$, giving the first result.

We obtain the second result as follows: let l_σ denote the length of $u_\sigma(\partial\Delta_m \cap D) \subset T^*M$. For $\rho > 0$ small enough we can find m disjoint solid balls of radius $\rho\sigma$ whose centers lie on $u_\sigma(\partial\Delta_m \cap D_\sigma)$, where $m = l_\sigma/2\sigma$ rounded down. Lemma A.3 then shows that:

$$\text{area}(D_\sigma) \geq \left(\frac{l_\sigma}{2\sigma}\right) (C(\rho\sigma)^2) = \frac{1}{2}Cl_\sigma\rho^2\sigma$$

Since $\text{area}(D_\sigma) \leq C\sigma$, this shows that l_σ must have some maximum.

Lemma A.6. ([4], Theorem 9.4) Fix $q \geq 1$ and $\delta_{k-1} > \delta_k \geq 0$. Let A be the open unit disk in \mathbb{C} or the half disk with boundary on \mathbb{R} . For any compact $K \subset A$, there exists a constant C such that for all holomorphic maps $u \in W^{k,2+\delta_{k-1}}(A, \mathbb{C}^n)$, we have:

$$\|u\|_{W^{k+1,2+\delta_k}(K)} \leq C \|u\|_{W^{k,2+\delta_k}(A)} \quad (\text{A.10})$$

Proof: See [4], Theorem 9.4.

Define G_r to be the open r -disk centered at the origin in \mathbb{C} , and $H_r = G_r \cap \{x \geq 0\}$.

There exists $d > 0$ such that, if $p, q \in \Sigma_1$, and $\pi_M(p) = \pi_M(q) \in \pi_M(\Sigma_2)$, then the distance between p, q is greater than d when $\sigma = 1$. Let $S_d \subset \Sigma_1$ be an arbitrary open set with diameter less than d . Choose $\delta > 0$ small enough that S_d has a tubular neighborhood of radius δ . Then let $N(\Pi(S_d), \delta)$ denote the restriction to S_d of a δ -tubular neighborhood in the product neighborhood. Define $V(S_d, \delta) \subset \Lambda_\sigma$ to be the connected segment of $\pi_M^{-1}(N(\Pi(S_d), \delta))$ containing S .

Recall from section A.5 that $U(k, \delta)$ is the product neighborhood of radius δ around Σ_k , and that $V(k, \delta) = U(k, \delta) - U(k+1, \epsilon_0)$.

Lemma A.7. ([6], Lemma 5.6) There exists $C > 0$ such that if $u : G_2 \rightarrow T^*M$ is pseudo-holomorphic then:

$$\sup_{G_1} |Du| \leq C \|Du\|_{L^2, G_2}$$

In addition, for all $K > 0$ large enough there exists $C > 0$ such that if $u : (H_2, \partial H_2) \rightarrow (T^*M, \pi_{\mathbb{C}}(\Lambda_\sigma))$ is pseudoholomorphic, and if $u(\partial H_2)$ lies outside a $K\sigma$ -neighborhood of $\pi_{\mathbb{C}}(\Sigma_1) \subset \pi_{\mathbb{C}}(\Lambda_\sigma)$, then:

$$\sup_{H_1} |Du| \leq C \|Du\|_{L^2, H_2}$$

Finally, for all $d > 0$ small enough, if $u(H_2, \partial H_2) \rightarrow (T^*M, V(S, \delta))$ is a pseudoholomorphic map such that $u(\partial H_2) \subset T^*(N(\Pi(S_d), \delta))$ then:

$$\sup_{H_1} |Du^T| \leq C \|Du^T\|_{L^2, H_2}$$

Proof: We begin with the first case. If $\|Du\|_{L^2, G_2} = \infty$ the inequality is trivially true, so we assume it does not. Recall from [3], Theorem 27.18, that the Sobolev inequality for $k > n/q$ is:

$$\|f\|_{C^{k-\lfloor n/p \rfloor - 1, \gamma}(U)} \leq C \|f\|_{W^{k,p}(U)}$$

Where n is the dimension of the space, U is a bounded open subset of \mathbb{R}^n with C^1 boundary, $C^{m,\gamma}(U)$ is a Holder space, and γ is defined by:

$$\gamma = \left\lfloor \frac{n}{p} \right\rfloor + 1 - \frac{n}{p}$$

Where $\lfloor x \rfloor$ denotes the floor of x . In our case $f = Du$, $U = G_1$ is a subset of $\mathbb{C} = \mathbb{R}^2$, so $n = 2$, and we choose $k = 1$. Therefore, for $q > 2$:

$$\|Du\|_{C^{0,\gamma}(G_1)} \leq C_1 \|Du\|_{W^{1,q}(G_1)}$$

Since $\|v\|_{C^{0,\gamma}(U)} \geq \|v\|_{C^0(U)}$ for any γ, U , this gives us:

$$\sup_{z \in G_1} |Du| \leq C_1 \|Du\|_{W^{1,q}(G_1)} \quad (\text{A.11})$$

Then, we apply Lemma A.6 to Du using $K = G_1, A = G_r, 1 < r < 2$. From this we obtain:

$$\|Du\|_{W^{1,q}(G_1)} \leq C_2(r) \|Du\|_{W^{0,q}(G_r)} = C_2(r) \|Du\|_{L^q(G_r)} \quad (\text{A.12})$$

Further, we have:

$$\begin{aligned} \|Du\|_{L^q(G_r)}^q &= \int_{G_r} |Du|^q \leq \left(\sup_{z \in G_r} |Du|^{q-2} \right) \left(\int_{G_r} |Du|^2 \right) \leq \left(\sup_{z \in G_r} |Du|^{q-2} \right) \left(\int_{G_2} |Du|^2 \right) \\ \|Du\|_{L^q(G_r)} &\leq \sup_{z \in G_r} |Du|^{(q-2)/q} \cdot \|Du\|_{L^2(G_2)}^{2/q} \end{aligned} \quad (\text{A.13})$$

Combining equations A.11, A.12, and A.13 we have:

$$\sup_{z \in G_1} |Du| \leq C_3(r) \|Du\|_{L^2(G_2)}^{2/q}$$

Where $C_3(r)$ is defined by:

$$C_3(r) = C_1 C_2(r) \sup_{z \in G_r} |Du|^{(q-2)/q} \quad (\text{A.14})$$

The limit of $C_2(r)$ as $r \rightarrow 1$ is given by:

$$\lim_{r \rightarrow 1} C_2(r) = \frac{\|Du\|_{W^{1,q}(G_1)}}{\|Du\|_{W^{0,q}(G_1)}} \quad (\text{A.15})$$

Combining equation A.14, equation A.15, and the fact that the supremum of $|Du|$ will be strictly declining as r shrinks, tells us that the limit of $C_3(r)$ as $r \rightarrow 1$ must be defined and finite. Therefore the same inequality holds with $r = 1$, giving us:

$$\sup_{z \in G_1} |Du| \leq C \|Du\|_{L^2(G_2)}$$

The process is precisely analogous for the other cases.

Recall from section A.5 that D_σ is the closed set in Δ_m made of vertical lines l such that $u(l)$ lies outside an arbitrarily small neighborhood of Σ_2 . Let $A_r(p)$ denote the points in D_σ which are connected to p by a path in D_σ of length at most r .

Lemma A.8. (*[6], Lemma 5.7*) *If ϵ_2 and σ are small enough, if $d_1, d_2 > 0$ are small enough, and if $C_1 > 0$ is big enough, then there exists $C = C(\epsilon_2) > 0$ such that:*

- *Let $p \in D_\sigma$, and at least $4d_2$ distance from ∂D_σ . Then:*

$$\sup_{B_{d_2}(p)} |Du| \leq C\sigma$$

- *If $p \in \partial \Delta_m \cap D_\sigma$ and $u(A_{4d_2}(p) \cap \partial \Delta_m)$ is outside of a $C_1\sigma$ -neighborhood of $\pi_{\mathbb{C}}(\Sigma_1) \subset \pi_{\mathbb{C}}(\Lambda_\sigma)$, then:*

$$\sup_{A_{d_2}(p)} |Du| \leq C\sigma$$

- *If $p \in \partial \Delta_m \cap D_\sigma$ and $u(A_{4d_2}(p) \cap \partial \Delta_m) \subset V(S, \delta)$ and $u(A_{4d_2}(p)) \subset T^*N(\Pi(S), \delta)$ then:*

$$\sup_{A_{d_2}(p)} |Du^T| \leq C\sigma$$

Proof: We begin with the second case. There is a biholomorphic map with uniformly bounded derivatives from $A_{4d_2}(p)$ to H_{4d_2} . Then, by Lemma A.7, there is a constant C' such that for all $z \in H_{2d_2}$, $|Du_\sigma(z)| \leq C' \|Du_\sigma\|_{L^2(H_{4d_2})}$, and:

$$\|Du_\sigma\|_{L^2(H_{4d_2})}^2 = \int_{H_{4d_2}} |Du_\sigma|^2 = \int_{H_{4d_2}} g(Du_\sigma, Du_\sigma) = \int_{H_{4d_2}} \omega(Du_\sigma, JDu_\sigma) = \int_{H_{4d_2}} u_\sigma^* \omega$$

Therefore $\|Du_\sigma\|_{L^2(H_{4d_2})}$ is the square root of the symplectic area of $u_\sigma(H_{4d_2}(p))$. Since $H_{4d_2}(p) \subset D_\sigma$, and the symplectic area of $u_\sigma(D_\sigma)$ is less than the action of the positive Reeb chords, and the action of the positive Reeb chords is scaled by σ , we obtain $|Du_\sigma(z)| \leq C''\sigma^{1/2}$ for all $z \in H_{2d_2}$.

We define a norm on the space of linear operators. For a given point p and a linear operator $J_p : T_p M \rightarrow T_p M$, we define:

$$|J_p| = \sup_{v \in T_p M, |v|=1} \{|Jv|\}$$

And for an operator $J : TM \rightarrow TM$, we define:

$$|J| = \sup_{p \in M} \{|J_p|\}$$

Pick a complex coordinate chart around $u_\sigma(p)$ that agrees with the ambient almost complex structure at $u_\sigma(p)$. We denote the complex structure from the coordinate chart by \tilde{J} . Then, for any $q \in B_{C''\sigma^{1/2}}(u(p))$, $|\tilde{J}_q - (J_\sigma)_q| = \mathcal{O}(\sigma^{1/2})$.

Let the sheet of $\pi_{\mathbb{C}}(\Lambda_\sigma)$ in T^*M containing $u_\sigma(p)$ be the graph $\Gamma_{\sigma\alpha}$ of a 1-form $\sigma\alpha$. We claim that for all σ small enough, we can find a diffeomorphism Θ such that:

- $\Theta(\Gamma_{\sigma\alpha})$ equals the 0-section, and is therefore real analytic
- $d\Theta + J_\sigma \circ d\Theta \circ \tilde{J} = 0$ along $\Gamma_{\sigma\alpha}$
- Let d_{C^1} be the sup norm metric on C^1 . $d_{C^1}(\Theta, \text{Id}) \leq \eta$ for arbitrarily small η .

We prove the existence of Θ by calculating its inverse. Let $z = x + iy$ be the coordinates of the local complex chart such that the 0-section corresponds to $\{y = 0\}$. Define $\psi(x) = (x, \sigma\alpha_x)$. Then, for very small y , define:

$$\Theta^{-1}(x, y) = \psi(x) + \sum_i (y_i \cdot J_{\psi(x)} \psi_* \partial_{x_i})$$

And extend the map arbitrarily to the rest of the chart. Then $\Theta^{-1}(x, 0) = \psi(x)$, so it sends the zero section to the graph $\Gamma_{\sigma\alpha}$, and it is C^1 -close to the identity. And, along the zero-section:

$$\Theta_*^{-1} \partial_{x_i} = \psi_* \partial_{x_i}$$

$$\Theta_*^{-1} \partial_{y_i} = (J_\sigma)_{\psi(x)} \psi_* \partial_{x_i}$$

So $d\Theta^{-1} + \tilde{J} \circ d\Theta^{-1} \circ J_\sigma = 0$. This gives us Θ .

Define $J_\Theta = \Theta_* \circ J_\sigma \circ \Theta_*^{-1}$. Then $|J_\Theta - \tilde{J}| = \mathcal{O}(\eta)$, and $\Theta \circ u$ is J_Θ -holomorphic.

Define $\hat{u}_\sigma = \sigma^{-1} \Theta \circ u_\sigma$ and $\hat{J}(x, y) = J_\Theta(\sigma x, \sigma y)$. Then $|\hat{J} - \tilde{J}| = \mathcal{O}(\eta)$ and \hat{u}_σ is \hat{J} -holomorphic. Furthermore, \hat{u}_σ is \tilde{J} -holomorphic on $\partial H_{4d_2}(p)$. We extend \hat{u}_σ by copying it over the 0-section and the boundary, obtaining a map $\tilde{u}_\sigma : D \rightarrow \mathbb{C}^n$, where D is a disc, \tilde{u}_σ is C^1 , and $d\tilde{u}_\sigma + \hat{J} \circ d\tilde{u}_\sigma \circ \tilde{J} = 0$, where \hat{J} is extended over the new area by $\hat{J}(z) = \hat{J}(\hat{u}(\bar{z}))$, where the bars denote the complex conjugate.

Now define a map $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ by:

$$F(z_1, \dots, z_n) = (e^{iz_1}, \dots, e^{iz_n})$$

And define $f_\sigma = F \circ \tilde{u}_\sigma$.

Let $\hat{y}_\sigma, \tilde{y}_\sigma$ be the imaginary components of $\hat{u}_\sigma, \tilde{u}_\sigma$. Since \hat{u}_σ is pseudoholomorphic, the function $|\hat{y}_\sigma|^2$ is subharmonic, and is therefore bounded. Since \tilde{u}_σ is the doubling of \hat{u}_σ , the same is true of $|\tilde{y}_\sigma|^2$. Therefore, $|f_\sigma|$ is uniformly bounded, and the derivatives of F are uniformly bounded in a neighborhood of the image of \tilde{u}_σ . Furthermore:

$$df_\sigma \circ \tilde{J} - J_F \circ df_\sigma = 0$$

Where $J_F = dF \circ \hat{J} \circ dF^{-1}$. Since $d = \partial_{\tilde{J}} + \bar{\partial}_{\tilde{J}}$, this implies:

$$\begin{aligned} -\partial_{\tilde{J}} f_\sigma + \bar{\partial}_{\tilde{J}} f_\sigma + J_F \circ (\partial_{\tilde{J}} f_\sigma - \bar{\partial}_{\tilde{J}} f_\sigma) &= 0 \\ (\tilde{J} + J_F) \circ \bar{\partial}_{\tilde{J}} f_\sigma + (\tilde{J} - J_F) \circ \partial_{\tilde{J}} f_\sigma &= 0 \end{aligned}$$

Define $q(z) = (\tilde{J} + J_F)^{-1}(\tilde{J} - J_F)$. This then becomes:

$$\bar{\partial}_{\tilde{J}} f_\sigma + q(z) \partial_{\tilde{J}} f_\sigma = 0$$

Since F is \tilde{J} -holomorphic:

$$|J_F - \tilde{J}| \leq |dF| |\hat{J} - \tilde{J}| |dF^{-1}| = \mathcal{O}(\eta)$$

We obtain:

$$q(z) = \mathcal{O}(\eta)$$

Now let $\beta : G_1 \rightarrow \mathbb{R}$ be a cutoff function which equals 1 on $G_{\frac{1}{2}}$ and equals 0 outside $G_{\frac{3}{4}}$, and define $f_\sigma^1 = \beta f_\sigma$. Then, define:

$$g_\sigma^1 = \bar{\partial}_{\tilde{J}} f_\sigma^1 + q(z) \partial_{\tilde{J}} f_\sigma^1 = \bar{\partial}_{\tilde{J}} (\beta f_\sigma) + q(z) \partial_{\tilde{J}} (\beta f_\sigma) = (\bar{\partial}_{\tilde{J}} \beta + q(z) \partial_{\tilde{J}} \beta) f_\sigma$$

Since β is compactly supported, and therefore its derivatives are bounded, and since $q(z) = \mathcal{O}(\eta)$, we conclude $|g_\sigma^1| \leq C|f_\sigma|$. Therefore:

$$|\bar{\partial}_{\tilde{J}} f_\sigma^1| \leq C_0 \eta |\partial_{\tilde{J}} f_\sigma^1| + C_1 |f_\sigma^1| \quad (\text{A.16})$$

Provided η is small enough, we can rearrange equation A.16 to obtain:

$$2|\bar{\partial}_{\tilde{J}} f_\sigma^1|^2 \leq C'_1 |f_\sigma|^2$$

Then add $|\partial_{\tilde{J}} f_\sigma^1|^2 - |\bar{\partial}_{\tilde{J}} f_\sigma^1|^2$ to both sides to obtain:

$$|\partial_{\tilde{J}} f_\sigma^1|^2 + |\bar{\partial}_{\tilde{J}} f_\sigma^1|^2 \leq (|\partial_{\tilde{J}} f_\sigma^1|^2 - |\bar{\partial}_{\tilde{J}} f_\sigma^1|^2) + C'_1 |f_\sigma|^2 \quad (\text{A.17})$$

Since ω is exact, by Stokes theorem:

$$\int (|\partial_{\tilde{J}} f_\sigma^1|^2 - |\bar{\partial}_{\tilde{J}} f_\sigma^1|^2) dA = \int (f_\sigma^1)^* \omega = 0$$

Therefore, if we integrate both sides of equation A.17, we obtain:

$$\|Df_\sigma^1\|_{L^2}^2 \leq C \sup |f_\sigma|^2$$

Which is less than or equal to some constant K' since f_σ is bounded. Note that, since the derivatives of F are uniformly bounded on a neighborhood of the image of \tilde{u}_σ , we have:

$$\|D\hat{u}_\sigma\|_{L^2} \leq C \|Df_\sigma^1\|_{L^2}$$

Therefore $\|D\hat{u}_\sigma\|_{L^2}$ is bounded by some constant, and therefore, by scaling, $\|Du_\sigma\|_{L^2}^2 = \mathcal{O}(\sigma^2)$. Combining this with Lemma A.8, we obtain the result. The first and third part of the lemma follow analogously.

A.6 Subdivision of the Domain

Let $N(k, d) = \pi_M(U(k, d))$. Pick $\delta > 0$. Let b_i^c denote the points in $\partial\Delta_m$ such that $u(b_i^c) \in \partial(T^*N(1, c\delta))$ and $u(\partial\Delta_m)$ crosses $\partial T^*N(1, c\delta)$ at b_i^c , ordered by position on the boundary. Puncture the disk Δ_m at every b_j^c, b_{j+1}^c such that there is b_k^{c-1} so that $b_j^c < b_k^{c-1} < b_{j+1}^c$ for $c = 2$ or 4 . (This is a correction of the version of boundary puncturing that appears in [6]). This gives us a new disk with more punctures but the same map that we will denote Δ_r .

We define the **boundary minimum** of a component I of $\partial\Delta_r$ to be the point $(x, y) \in I$ with minimum x value.

In addition, recall from section A.5 that D_σ denotes the restriction of Δ_r to vertical lines contained in the pre-image of $T^*M - T^*U(2, \epsilon_3)$, where ϵ_3 is arbitrarily small.

Lemma A.9. ([6], Lemma 5.9) *There exists a constant $C = C(\delta) > 0$ that does not depend on σ such that the number of added punctures is less than or equal to C for fixed δ and any σ .*

Proof: Each added puncture of Δ_r corresponds to b_j^c such that there exists b_k^{c-1} so that $b_j^c < b_k^{c-1} < b_{j+1}^c$ on $\partial\Delta_m$ for $c = 2, 3, 4$. The length of the segment between b_k^{c-1}, b_{j+1}^c is bounded from below by δ , and by lemma A.5 the length of $u(\partial\Delta_r \cap D_\sigma)$ is bounded above for any σ , so the result follows.

We now label the boundary components of Δ_r by the following types:

out: $u(I) \subset T^*(M - N(1, 3\delta))$
0: $u(I) \subset T^*(N(1, 4\delta) - N(1, \delta))$
in: $u(I) \subset T^*N(1, 2\delta)$

Then, for I a boundary component and $0 < \rho < \frac{1}{4}$, define $N_\rho(I)$ to be a ρ -neighborhood of I in Δ_m , and:

$$\Omega_\rho = D_\sigma - \bigcup_{I \subset \partial\Delta_r} N_\rho(I)$$

Fix a small $\epsilon > 0$.

Lemma A.10. ([6], Lemma 5.10) *There exists C such that if $\sigma > 0$ is small enough, then:*

$$\sup_{z \in \Theta_\epsilon} |Du(z)| \leq C\sigma$$

$$\Theta_\epsilon = \Omega_\epsilon \cup \left(\bigcup_{I \in \text{out} \cup 0} N_\epsilon(I) \right)$$

Proof: This is a direct consequence of lemma A.8.

Let $D'_\sigma \subset D_\sigma$ be the subset containing all vertical line segments in D_σ which connect a point in a boundary component of type **in** to some other boundary point. Note $\partial D'_\sigma - \partial D_\sigma$ is a collection of vertical line segments.

Lemma A.11. ([6], Lemma 5.11) *For any $0 < a < 1$ and sufficiently small $\sigma > 0$, the distance from any $p \in I$, where I is of type **out**, to D'_σ is larger than σ^{-a} . In particular, if l is a vertical line segment in $\partial D'_\sigma - \partial D_\sigma$ and $q \in \partial l$ then q is either a boundary minimum on a segment of type **in** or it lies on a boundary segment of type **0**.*

Proof: Suppose $p \in I$, I of type **out**, and the distance from p to D'_σ is less than σ^{-a} . Then there exists a path in $\Omega_\epsilon \cup N_\epsilon(I) \subset D_\sigma$ of length less than $\sigma^{-a} + 5r$, where r is the number of punctures of Δ_r , from p to a point q midway between two horizontal boundary segments of length 1, at least one of which is of type **in**. Call the segment of type **in** I_1 and the other I_2 .

Since $|Du_\sigma| = \mathcal{O}(\sigma)$ along this path and $u_\sigma(p) \in T^*(M - N(1, 3\delta))$, we know that:

$$u_\sigma(q) \in T^*(M - N(1, 3\delta - (5r\sigma + \sigma^{1-a})))$$

Therefore, for σ small enough, $u_\sigma(q) \in T^*(M - N(1, \frac{5}{2}\delta))$. Let $I_3 \subset \Delta_r$ be a horizontal segment of length 1 that intersects q . For σ small enough, $u_\sigma(I_3) \subset T^*(M - N(1, \frac{5}{2}\delta))$. By definition, $u_\sigma(I_1) \subset T^*N(1, 2\delta)$. Call the region between I_1 and I_3 R , and observe that $R = [0, 1] \times [0, \frac{1}{2}]$. Observe further that the image under u_σ of every vertical path in R has length at least $\frac{1}{2}\delta$. Therefore, if t is the vertical coordinate of Δ_r :

$$\iint_R \left| \frac{\partial u_\sigma}{\partial t} \right|^2 dA \geq \frac{1}{2\delta}$$

From this, we conclude that the L^2 -norm of $|Du|$ is bounded below. This contradicts lemma A.5, which states that the symplectic area of $u(D_\sigma)$ is bounded above by $C\sigma$ for some constant C .

Let F_l be the region of points of distance l or less from D'_σ . Choose ρ' so that $\log(\sigma^{-1}) \leq \rho' \leq 2\log(\sigma^{-1})$ and $\partial F_{\rho'} - \partial D_\sigma, \partial F_{\frac{1}{2}\rho'} - \partial D_\sigma$ are vertical line segments disjoint from boundary minima. Define $D'_1(\sigma) = F_{\rho'}, D_0(\sigma) = D_\sigma - (F_{\frac{1}{2}\rho'})$. Note that if $p \in \partial D_0 \cap \partial D_\sigma$ then p is in a boundary component of type **0** or **out**, and if $p \in \partial D'_1 \cap \partial D_\sigma$ then p is in a boundary component of type **0** or **in**. Lemma A.10 implies that:

$$\sup_{D_0} |Du_\sigma| \leq C\sigma$$

Lemma A.12. ([6], Lemma 5.12) For small enough σ , $u(D'_1(\sigma)) \subset T^*U(1, \frac{9}{2}\delta)$ and $u(D_0(\sigma)) \subset T^*(M - N(1, \frac{1}{2}\delta))$.

Proof: Let $q \in D_0(\sigma)$. Then q is linked by a path γ of length less than $5r$, where r is the number of punctures of Δ_r , to a point $p \in \partial D_\sigma \cap \partial D_0(\sigma)$. p must lie in a component of type **0** or **out**, so $u_\sigma(p) \in T^*(M - N(1, \delta))$. Since $|Du_\sigma| \leq C\sigma$ along the path, for σ small enough the second statement follows.

$\partial D'_1(\sigma)$ consists of boundary segments of types **0** and **in** and vertical lines ending on boundary components of type **0**. We know from the definition that the image of the boundary segments will lie in $T^*N(1, 4\delta)$, while the bound on $|Du_\sigma|$ allows us to ensure that $u(\partial D_1(\sigma)) \subset T^*N(1, \frac{17}{4}\delta)$ for σ small enough. Then, if $u(D'_1(\sigma))$ does not lie inside $T^*N(1, \frac{9}{2}\delta)$, we may bound its area from below for any σ by Lemma A.3, our modified monotonicity lemma.

We can then repeat this process: we add additional punctures at the intersections of $u(\partial \Delta_r)$ with $T^*N(2, c\delta)$, $c = 1, 2, 3, 4$. We label the boundary components of $D'_1(\sigma)$ with **out'**, **0'**, **in'**. We bound u_σ^T in a neighborhood Θ'_ϵ , and use equivalents of Lemmas 5.13 and 5.14 for u_σ^T to split $D'_1(\sigma)$ into $D_1(\sigma)$, which stays away from Σ_2 , and $D'_2(\sigma) = D_2(\sigma)$, which does not. We only do this once, unlike in [6]. We have thus divided $\Delta_r = D_0(\sigma) \cup D_1(\sigma) \cup D_2(\sigma)$, so that $D_0(\sigma)$ maps to a neighborhood away from the singularities, $D_1(\sigma)$ maps to a neighborhood of the cusp edges, and $D_2(\sigma)$ maps to a neighborhood of Σ_2 .

A.7 Convergence of Disk Boundaries to Flow Lines

Let $W_j(\sigma)$ be a neighborhood of the boundary minima of $D_j(\sigma)$ such that:

- $\partial W_j(\sigma)$ is a union of arcs in $\partial D_j(\sigma)$ and of vertical line segments;
- Each component of $W_j(\sigma)$ contains at least one boundary minimum;
- And the width of each component of $W_j(\sigma)$ is at most $\log \sigma^{-1}$.

Now consider a sequence:

$$u_\sigma : (\Delta_m, \partial \Delta_m) \rightarrow (T^*M, \pi_\mathbb{C}(\Lambda_\sigma)), \sigma \rightarrow 0$$

Lemma A.13. $u_\sigma(W_0(\sigma))$ converges to a point as $\sigma \rightarrow 0$.

Proof: By Lemma A.10, $|Du_\sigma| = \mathcal{O}(\sigma)$ in $W_0(\sigma)$, so the area of $u_\sigma(W_0(\sigma))$ is $\mathcal{O}(\sigma \log(\sigma^{-1}))$.

Let σ_0 be the value of σ for which the number of components of $D_0(\sigma) - W_0(\sigma)$ reaches its maximum. (Since the total number of added punctures is bounded by Lemma A.9, we must be able to select $W_0(\sigma)$ in such a way that a

maximum is eventually achieved.) Restrict σ to $\sigma < \sigma_0$, and consider a vertical line segment $l \subset D_0(\sigma) - W_0(\sigma)$. Let X be the component of $D_0(\sigma) - W_0(\sigma)$ containing l . If X is an infinite or half-infinite rectangle, it can be parameterized by $(-\infty, 0] \times [0, 1]$ or $[0, \infty) \times [0, 1]$ or $\mathbb{R} \times [0, 1]$. In each case, l is specified by a choice of x , so our choice of l is well-defined for any $\sigma < \sigma_0$. If X is a finite rectangle, it can be parameterized by $X = [0, d_\sigma] \times [0, 1]$, where d_σ depends on σ . Pick some value of σ , which we call $\hat{\sigma}$. Then, for $\sigma = \hat{\sigma}$, l is specified by a choice of $x \in [0, d_{\hat{\sigma}}]$. Then we can define a function $x_\sigma = (d_\sigma x / d_{\hat{\sigma}})$. This lets us consider our choice of l to be well-defined for any value of $\sigma < \sigma_0$. We can similarly show that a choice of vertical line segment $l \subset D_1(\sigma) - W_1(\sigma)$ is well-defined for any $\sigma < \sigma_0$.

Recall from section A.5 that we can parameterize u_σ by (q_σ, p_σ) , where q_σ is the point in the base space and p_σ is the cofiber coordinate.

Let $l \subset D_0(\sigma) - W_0(\sigma)$. By the definition of $D_0(\sigma)$, the image of l is outside of a neighborhood of Σ_1 . We can therefore find a neighborhood of the image of l in which $\pi_{\mathbb{C}}(\Lambda) \subset T^*M$ can be parameterized as the graph of some collection of functions $M \rightarrow T^*M$. We refer to these graphs as sheets, by analogy to the sheets of $\pi_F(\Lambda)$. Let b_1, b_0 be functions of the sheets containing the image of ∂l .

Lemma A.14. ([6], Lemma 5.13) *For all sufficiently small $\sigma > 0$, along any vertical line segment $l \subset D_0(\sigma) - W_0(\sigma)$:*

$$\frac{1}{\sigma} \nabla_i p_\sigma(0, t) - (b_1(q_\sigma(0, 0)) - b_0(q_\sigma(0, 0))) = \mathcal{O}(\sigma)$$

$$\frac{1}{\sigma} \nabla_\tau p_\sigma(0, t) = \mathcal{O}(\sigma)$$

Where ∇_* denotes the connection, the subscripts t, τ are used to indicate $\partial_t, \partial_\tau$, and t, τ are the vertical, horizontal coordinates respectively of $D_0(\sigma) - W_0(\sigma)$.

Proof: Let $\Theta_{c_\sigma} = [-c_\sigma, c_\sigma] \times [0, 1] \subset D_0(\sigma) - W_0(\sigma)$ be an arbitrarily small neighborhood around l , with $c_\sigma \leq \sigma \log(\sigma^{-1})$. By Lemma A.8, $|Du_\sigma| \leq C\sigma$ on Θ_{c_σ} . Therefore, we can pick a radius R such that $\pi_M(u_\sigma(\Theta_{c_\sigma})) \subset T^*D_{\sigma R}$, where $D_{\sigma R}$ is the geodesic disk in M of radius σR centered around $\pi_M(u_\sigma(0, 0))$. We think of T^*D_R as a subset of \mathbb{C}^n , $(T^*D_R, J_1) = (\{q + ip : |q| \leq R\}, i)$. Let g_0 be the flat metric on D_R , and let J_1 be the corresponding standard complex structure on T^*D_R , with the coordinates chosen such that $(J_1)_{u_\sigma(0, 0)} = (J_\sigma)_{u_\sigma(0, 0)}$ and $u_\sigma(0, 0) = 0$. Recall that q_σ is the base space component and p_σ is the cofiber component of u_σ ; there exists some K such that $|p_\sigma(\Theta_{c_\sigma})| \leq K\sigma$ for all σ . Define U_σ to be $\{q + ip : |q| \leq R\sigma, |p| \leq K\sigma\}$.

Define an almost complex structure $(\hat{J}_\sigma)_{(q, p)} = (J_\sigma)_{(\sigma q, \sigma p)}$, and define $\hat{u}_\sigma = \sigma^{-1} u_\sigma : \Theta_{c_\sigma} \rightarrow \mathbb{C}^n$. \hat{u}_σ is \hat{J}_σ -holomorphic.

We claim that $|\hat{J}_\sigma - J_1|_{C^2(U_\sigma)} = \mathcal{O}(\sigma)$. We will prove the statement in dimension 2; the proof can be extended to higher dimensions by adding appropriate

summation signs. Define f_1, f_2 by:

$$(J_\sigma)_z \partial_q = f_1(z, \sigma) \partial_q + f_2(z, \sigma) \partial_p$$

We know that $f_1(0, \sigma) = 0, f_2(0, \sigma) = 1$ because $(J_1)_{u_\sigma(0,0)} = (J_\sigma)_{u_\sigma(0,0)} = (\hat{J})_{(0,0)}$. We will ordinarily omit the σ coordinate of f_1, f_2 . Since U_σ is compact, $|f'_1(\sigma z)|, |f'_2(\sigma z)|, |f''_1(\sigma z)|, |f''_2(\sigma z)|$ all obtain some maximum; let K_1 be some number greater than all of them.

Using the Taylor expansion, we obtain:

$$|(\hat{J}_\sigma)_z(\partial_q) - (J_1)_z(\partial_q)|^2 = (f_1(\sigma z))^2 + (1 - f_2(\sigma z))^2 = \mathcal{O}(\sigma^2)$$

Therefore $|(\hat{J}_\sigma)_z(\partial_q) - (J_1)_z(\partial_q)| = \mathcal{O}(\sigma)$. From this, we obtain:

$$D \left(|(\hat{J}_\sigma)_z(\partial_q) - (J_1)_z(\partial_q)|^2 \right) = 2\sigma f_1(\sigma z) f'_1(\sigma z) - 2\sigma(1 - f_2(\sigma z)) f'_2(\sigma z)$$

$$\left(D |(\hat{J}_\sigma)_z(\partial_q) - (J_1)_z(\partial_q)| \right) \left(|(\hat{J}_\sigma)_z(\partial_q) - (J_1)_z(\partial_q)| \right) \leq$$

$$K_1 \sigma f_1(\sigma z) - K_1 \sigma(1 - f_2(\sigma z)) = \mathcal{O}(\sigma^2)$$

Therefore $D |(\hat{J}_\sigma)_z(\partial_q) - (J_1)_z(\partial_q)| = \mathcal{O}(\sigma)$. From this we further obtain:

$$D^2 \left(|(\hat{J}_\sigma)_z(\partial_q) - (J_1)_z(\partial_q)|^2 \right) = 2\sigma^2 f_1(\sigma z) f''_1(\sigma z) +$$

$$2\sigma^2 (f'_1(\sigma z))^2 - 2\sigma^2 (1 - f_2(\sigma z)) f''_2(\sigma z) + 2\sigma^2 (f'_2(\sigma z))^2$$

$$\left(D^2 |(\hat{J}_\sigma)_z(\partial_q) - (J_1)_z(\partial_q)| \right) \left(|(\hat{J}_\sigma)_z(\partial_q) - (J_1)_z(\partial_q)| \right) + \left(D |(\hat{J}_\sigma)_z(\partial_q) - (J_1)_z(\partial_q)| \right)^2 \leq$$

$$\sigma^2 K_1 f_1(\sigma z) + \sigma^2 K_1^2 - \sigma^2 (1 - f_2(\sigma z)) K_1 + \sigma^2 K_1^2$$

$$\left(D^2 |(\hat{J}_\sigma)_z(\partial_q) - (J_1)_z(\partial_q)| \right) \left(|(\hat{J}_\sigma)_z(\partial_q) - (J_1)_z(\partial_q)| \right) + \mathcal{O}(\sigma^2) \leq \mathcal{O}(\sigma^2)$$

Therefore $D^2 |(\hat{J}_\sigma)_z(\partial_q) - (J_1)_z(\partial_q)| \leq \mathcal{O}(\sigma^2)$. Combining these, we obtain:

$$|(\hat{J}_\sigma)(\partial_q) - (J_1)(\partial_q)|_{C^2(U_\sigma)} = \mathcal{O}(\sigma)$$

We can then repeat this calculation for ∂_p . Combining the two, we obtain $|\hat{J}_\sigma - J_1|_{C^2} = \mathcal{O}(\sigma)$. This calculation can be extended to higher dimension in a straight-forward manner by adding appropriate summation signs.

Recall that we defined the functions $\sigma b_0, \sigma b_1 : D_R \rightarrow T^* D_R$ to be the two sheets of $\pi_{\mathbb{C}}(\Lambda_\sigma)$ over D_R corresponding to the restriction of $u_\sigma|_{\partial\Theta_c}$ to $D_R \subset \mathbb{C}^n$. After scaling \mathbb{C}^n by σ^{-1} , we replace $x \rightarrow \sigma b_i(x)$ with $x \rightarrow b_i(\sigma x)$. Define $L_i = \{x + iy : y = b_i(0)\} \subset \mathbb{C}^n$; note that these are Lagrangian subspaces. Observe that:

$$\hat{p}_\sigma(\tau + i) - b_1(q(0 + i)) = b_1(\sigma q(\tau + i)) - b_1(\sigma q(0 + i)) \quad (\text{A.18})$$

In the next step, we will assume again that we are working in two dimensions without loss of generality. We can rewrite the righthand side of equation A.18 using its Taylor expansion around $\sigma q(0+i)$ to obtain:

$$\hat{p}_\sigma(\tau+i) - b_1(q(0+i)) = b'_1(\sigma q(0+i))\sigma(q(\tau+i) - q(0+i)) + \mathcal{O}(\sigma^2)$$

Therefore, for any τ , $\hat{p}_\sigma(\tau+i) - b_1(q(0+i)) = \mathcal{O}(\sigma)$, as does its derivative and double derivative by τ . We can repeat this calculation for $\hat{p}_\sigma(\tau+0i) - b_0(q(0+0i))$. Therefore there exists a function $f_\sigma : \Theta_c \rightarrow \mathbb{C}^n$ such that:

$$f_\sigma(0,0) = 0$$

$$\sup_{\Theta_c} |D^k f_\sigma| = \mathcal{O}(\sigma) \text{ for } k = 1, 2, 3$$

$$\hat{u}_\sigma(\tau+0i) + f_\sigma(\tau+0i) \in L_0$$

$$\hat{u}_\sigma(\tau+i) + f_\sigma(\tau+i) \in L_1$$

And $\hat{u}_\sigma + f_\sigma$ is J_1 -holomorphic to first order on $\partial\Theta_c$, that is, $\bar{\partial}_{J_1}(\hat{u}_\sigma + f_\sigma) = \mathcal{O}(\sigma)$. Define $u_\sigma^1 = \hat{u}_\sigma + f_\sigma : \Theta_c \rightarrow \mathbb{C}^n$, and define $u_\sigma^0 : \Theta_c \rightarrow \mathbb{C}^n$ to be the linear solution to:

$$\bar{\partial}_{J_1} u_\sigma^0 = 0$$

$$u_\sigma^0(\tau+0i) \in L_0$$

$$u_\sigma^0(\tau+i) \in L_1$$

$$u_\sigma^0(\tau+it) = ((b_1 - b_0)\tau, (1-t)b_0 + tb_1)$$

(There is a typo in the definition of u_σ^0 in [6], and we believe this is what he means.)

Define $v_\sigma = u_\sigma^1 - u_\sigma^0 : \Theta_c \rightarrow \mathbb{C}^n$. Then $v_\sigma(\partial\Theta_c) \subset \mathbb{R}^n$, v_σ is J_1 -holomorphic to first order on $\partial\Theta_c$, $v_\sigma(0,0) = 0$, and:

$$\sup_{\Theta_c} |D^k (\bar{\partial}_{J_1} v_\sigma)| = \mathcal{O}(\sigma), k = 0, 1, 2$$

Let $\mathcal{H}_{k,p,-\gamma}(\mathbb{R} \times [0,1], \mathbb{C}^n)$ denote the Hilbert space with the weight function $w(\tau) = e^{-\gamma}$ for $|\tau| \leq 1$ and $w(\tau) = e^{-\gamma|\tau|}$ for $|\tau| \geq 1$. Let $-\gamma = -3$. Define $\mathcal{H}_{3,2,-\gamma}(\mathbb{R} \times [0,1], \mathbb{C}^n; \mathbb{R}^n, 0_2)$ to be the space of functions F with boundary on \mathbb{R}^n and with three derivatives in the weighted Sobolev space L^2 , and such that the restriction to the boundary of $\bar{\partial}_{J_1} F$ and its first derivatives vanishes. We similarly define $\mathcal{H}_{2,2,-\gamma}(\mathbb{C}^n; 0_1)$ to be the space of functions with 2 derivatives in the weighted Sobolev space L^2 and which vanish to first order along the boundary. As shown in [4], Prop. 6.3, $\bar{\partial}_{J_1} : \mathcal{H}_{3,-\gamma}(\mathbb{R} \times [0,1], \mathbb{C}^n; \mathbb{R}^n, 0_2) \rightarrow \mathcal{H}_{2,-\gamma}(\mathbb{C}^n; 0_1)$ is a Fredholm operator of index n with kernel spanned by the constant functions. We will generally write $\bar{\partial}_{J_1} = \bar{\partial}$.

Let $W \subset \mathcal{H}_{3,-\gamma}(\mathbb{R} \times [0,1], \mathbb{C}^n; \mathbb{R}^n, 0_2)$ be the subspace of non-constant functions F with $F(0) = 0$. Then there exists a constant C such that:

$$\|w\|_{\mathcal{H}_{3,2,-\gamma}} \leq C \|\bar{\partial} w\|_{\mathcal{H}_{2,2,-\gamma}}$$

Now let $B : \Theta_c \rightarrow \mathbb{C}$ be a cutoff function such that:

- B is real-valued and holomorphic to first order on $\partial\Theta_{c_\sigma}$
- $B(z) = 1$ on $[-\frac{1}{4}c_\sigma, \frac{1}{4}c_\sigma] \times [0, 1]$
- $B(z) = 0$ outside $[-\frac{1}{2}c_\sigma, \frac{1}{2}c_\sigma] \times [0, 1]$

Then:

$$\|\bar{\partial}(Bv_\sigma)\|_{\mathcal{H}_{2,2,-\gamma}}^2 = \int \int_{\mathbb{R} \times [0,1]} (|\bar{\partial}v_\sigma|^2 + |D\bar{\partial}v_\sigma|^2 + |D^2\bar{\partial}v_\sigma|^2)$$

We split this into cases. First, observe that:

$$\begin{aligned} & \int_0^1 \int_{-\infty}^{\infty} |\bar{\partial}(Bv_\sigma)|^2 w(\tau) d\tau dt = \left\| \bar{\partial}(Bv_\sigma) \Big|_{[0,1] \times [-\frac{1}{4}c, \frac{1}{4}c]} \right\|_{H_{2,2,-\gamma}}^2 \\ & + \int_0^1 \int_{|\tau| > \frac{1}{4}c} |(\bar{\partial}v_\sigma)|^2 |B(\tau)|^2 e^{-\gamma|\tau|} d\tau dt + \int_0^1 \int_{|\tau| > \frac{1}{4}c} |(\bar{\partial}B)|^2 |v_\sigma|^2 e^{-\gamma|\tau|} d\tau dt \end{aligned}$$

Given that $B(\tau) = 1$ for $|\tau| \leq \frac{1}{4}c_\sigma$, the first term is equal to:

$$\left\| \bar{\partial}(Bv_\sigma) \Big|_{[0,1] \times [-\frac{1}{4}c, \frac{1}{4}c]} \right\|_{H_{2,2,-\gamma}}^2 = \left\| \bar{\partial}v_\sigma \Big|_{[0,1] \times [-\frac{1}{4}c, \frac{1}{4}c]} \right\|_{H_{2,2,-\gamma}}^2$$

Since $|\bar{\partial}v_\sigma| = \mathcal{O}(\sigma)$, this tells us that the first term of $\|\bar{\partial}Bv_\sigma\|_{\mathcal{H}_{2,2,-\gamma}}^2$ equals $\mathcal{O}(\sigma^2)$.

The second term, we calculate similarly; the bound on $|\bar{\partial}v|$ tells us that it also equals $\mathcal{O}(\sigma^2)$, and then no other term depends on t , giving us:

$$\int_0^1 \int_{|\tau| > \frac{1}{4}c} |(\bar{\partial}v_\sigma)|^2 |B(\tau)|^2 e^{-\gamma|\tau|} d\tau dt \leq 2K_0\sigma^2 \int_1^\infty |B(\tau)|^2 e^{-\gamma|\tau|} d\tau = \mathcal{O}(\sigma^2)$$

Finally, the third term we calculate as follows: since $|Du_\sigma|, |D^2u_\sigma|$ are bounded, $|Dv_\sigma|, |D^2v_\sigma|$ are bounded, and thus $|v_\sigma|, |Dv_\sigma| = \mathcal{O}(|c_\sigma|)$. Therefore:

$$\int_0^1 \int_{|\tau| > \frac{1}{4}c} |(\bar{\partial}B)|^2 |v_\sigma|^2 e^{-2\gamma|\tau|} d\tau dt = 2K_1|c|^2 \int_{\frac{1}{4}c_\sigma}^\infty e^{-2\gamma\tau} d\tau = -\frac{2K_1c^2e^{-\frac{1}{2}c_\sigma\gamma}}{2\gamma}$$

This equals $\mathcal{O}(c^2e^{-\frac{1}{2}\gamma c})$. Since $c_\sigma \leq \sigma \log(\sigma^{-1})$, this equals $\mathcal{O}(\sigma^2(\log(\sigma^{-1}))^2\sigma^{3/2}e^{-\frac{3}{2}\sigma}) = \mathcal{O}(\sigma^{7/2}\log(\sigma^{-1})) \leq \mathcal{O}(\sigma^2)$.

We can then repeat this process for $B(Dv_\sigma), B(D^2v_\sigma)$ to show that:

$$\|Bv_\sigma\|_{H_{2,3,-\gamma}} = \mathcal{O}(\sigma)$$

Since this controls the supremum norm over $[-\frac{1}{4}c_\sigma, \frac{1}{4}c_\sigma] \times [0, 1]$, we obtain:

$$\sup_{[0,1] \times [-\frac{1}{4}c_\sigma, \frac{1}{4}c_\sigma]} |D^k v_\sigma| = \mathcal{O}(\sigma), k = 0, 1$$

If we write $\hat{u}_\sigma(\tau, t) = (\hat{q}_\sigma(\tau, t), \hat{p}_\sigma(\tau, t))$, we obtain:

$$\begin{aligned}\nabla_t \hat{p}_\sigma(0, t) &= \frac{\partial \hat{p}_\sigma}{\partial t}(0, t) + \Gamma(\hat{q}_\sigma(0, t)) \left(\frac{\partial \hat{q}_\sigma}{\partial t}, \hat{p}_\sigma \right) \\ \nabla_\tau \hat{p}_\sigma(0, t) &= \frac{\partial \hat{p}_\sigma}{\partial \tau}(0, t) + \Gamma(\hat{q}_\sigma(0, t)) \left(\frac{\partial \hat{q}_\sigma}{\partial \tau}, \hat{p}_\sigma \right)\end{aligned}$$

Where Γ denotes the linear operator in $\nabla_u v = D_u v + \Gamma(u, v)$.

Recall that $\hat{u}_\sigma = \sigma^{-1} u_\sigma$. Further, $\hat{u}_\sigma - u_\sigma^0 = v_\sigma - f_\sigma$, where $\sup |D^k f_\sigma| = \mathcal{O}(\sigma)$ for $k = 1, 2, 3$, and $\frac{\partial p_\sigma^0}{\partial \tau} = \frac{\partial q_\sigma^0}{\partial t} = 0$, $\frac{\partial p_\sigma^0}{\partial t} = b_1(0) - b_0(0)$. Therefore:

$$\begin{aligned}\frac{\partial \hat{p}_\sigma}{\partial t} &= \frac{\partial p_\sigma^0}{\partial t} + \mathcal{O}(\sigma) = (b_1(0) - b_0(0)) + \mathcal{O}(\sigma) \\ \frac{\partial \hat{p}_\sigma}{\partial \tau} &= \mathcal{O}(\sigma)\end{aligned}$$

Lemma A.15. $u_\sigma^T(W_1(\sigma))$ converges to a point as $\sigma \rightarrow 0$.

Lemma A.16. For all sufficiently small $\sigma > 0$, then along any vertical line segment $l \subset D_1(\sigma) - W_1(\sigma)$:

$$\begin{aligned}\frac{1}{\sigma} \nabla_t p_\sigma^T(0, t) - (b_1(q_\sigma^T(0, 0)) - b_0(q_\sigma^T(0, 0))) &= \mathcal{O}(\sigma) \\ \frac{1}{\sigma} \nabla_\tau p_\sigma^T(0, t) &= \mathcal{O}(\sigma)\end{aligned}$$

Proof of Lemmas A.15 and A.16: The proofs are precisely analogous to the proofs of Lemmas A.13 and A.14, but with u_σ replaced with u_σ^T ; the base space replaced with $\pi(\Sigma_1)$; b_i replaced by their restriction to $T^\pi(\Sigma_1)$; Λ_σ replaced with its projection to $T^*\pi(\Sigma_1)$; and J_σ replaced by its restriction to $T^*\pi(\Sigma_1)$.

B Proof of Morse Lemmas

In this section we prove 3.1, 3.2, 3.3, and 3.4. First, recall that we define:

$$\mathcal{F}(f_1, \delta) = \{f_2 : M \rightarrow \mathbb{R} | f_2 \text{ is Morse, } |f_1 - f_2|_{C^1} < \delta\}$$

In addition, we need the well-known tubular flow theorem:

Theorem B.1 (Tubular Flow Theorem). *Let V be a vector field on an n -dimensional manifold M . Then, at any point $p \in M$ such that $V_p \neq 0$, there exists a neighborhood U of p and a diffeomorphism $\Psi : (-1, 1)^n \rightarrow U$ such that the trajectories of V in U are mapped to the trajectories of ∂_{x_1} in $(-1, 1)^n$.*

Proof of B.1: See [11], Theorem 2.1.1.

Then:

Lemma 3.1: *Let $f_1 : M \rightarrow \mathbb{R}$ be a Morse function, where M is a compact manifold. For any $\epsilon > 0$, there exists $\delta > 0$ such that, for any $f_2 \in \mathcal{F}(f_1, \delta)$, there is a bijection between the critical points of f_1 and f_2 , the ascending manifold of every critical point of f_2 lies within ϵ of the ascending manifold for the corresponding critical point of f_1 , and the descending manifold of every critical point of f_2 lies within ϵ of the descending manifold of the corresponding critical point of f_1 .*

Proof: It is well known that, if f_1 is Morse, then there exists δ' such that all $f_2 \in \mathcal{F}(f_1, \delta')$ have the same number of critical points as f_1 , and each critical point of f_2 has the same Morse index as the corresponding critical point of f_1 . Restrict δ to $\delta < \delta'$.

Now, since f_1 is smooth, $|\nabla f_1| : M \rightarrow \mathbb{R}$ is continuous, and $|\nabla f_1|^{-1}(0)$ is equal to the set of critical points of f_1 . Further, we know that:

$$\sup_{p \in M} ||\nabla f_1(p)| - |\nabla f_2(p)|| \leq |\nabla f_1 - \nabla f_2|_{C^0} \leq |f_1 - f_2|_{C^1}$$

Therefore, if $f_2 \in \mathcal{F}(f_1, \delta)$, then $|\nabla f_2|^{-1}(0) \subset |\nabla f_1|^{-1}(0, \delta)$. Therefore, if $\delta > 0$ is small enough, the critical points of f_2 will lie arbitrarily close to the corresponding critical points of f_1 . Let p_1, \dots, p_m denote the critical points of f_1 , and let p'_i denote the critical point of f_2 corresponding to p_i . Pick open ball-shaped Morse neighborhoods $\psi_i : N_i \rightarrow M$ for every critical point p_i , and assume that δ is small enough that $p'_i \in \psi_i(N_i)$ for every $f_2 \in \mathcal{F}(f_1, \delta)$ and every critical point p'_i . Define:

$$Q = M - \bigcup_i \psi_i(N_i)$$

Observe that Q is compact.

By the Tubular Flow Theorem, for every point p that is not a critical point, we can find a radius $\rho(p)$ such that there is an open neighborhood of p of radius $\rho(p) < \frac{1}{2}\epsilon$ that is diffeomorphic to $(-1, 1)^n$, and such that the diffeomorphism carries flow lines of ∇f_1 to flow lines of ∂_{x_1} . Let $U(p)$ denote the tubular flow neighborhood of p , and observe that $U(p)$ is an infinite open cover of the compact submanifold Q . We can therefore find a finite subcover U_1, \dots, U_m . Let $\phi_1 : [-1, 1]^n \rightarrow U_1, \dots, \phi_m : [-1, 1]^n \rightarrow U_m$ denote the diffeomorphisms. Equip



Figure 9: Open Cover of γ

each copy of $[-1, 1]^n$ with the pullback of the metric of M , rather than the standard Euclidean metric.

Pick an arbitrary flow line $\gamma : [0, l] \rightarrow M$ parameterized by arc length, so that $\gamma(0) = p_i, \gamma(l) = p_j$. Assume without loss of generality that $\psi_i(N_i), U_1, \dots, U_k, \psi_j(N_j)$ form an open cover of the image of γ (see Figure 9).

Let q_0 be the point where the pullback of γ to N_i intersects ∂N_i . $\psi_i^{-1}(U_1)$ will be open in N_i , so there is some disk of radius ϵ_0 around q_0 in ∂N_i which is contained within $\partial N_i \cap \psi_i^{-1}(U_1)$. Define the set:

$$S = \{x \in N_i | x_{k_i+1}^2 + \dots + x_n^2 < (\epsilon_0)^2\} \quad (\text{B.1})$$

Where k_i is the Morse index of p_i . The boundary of S can be considered as the union of two (possibly disconnected, possibly empty) overlapping parts:

$$V = \{x \in N_i | x_{k_i+1}^2 + \dots + x_n^2 = (\epsilon_0)^2\} \quad (\text{B.2})$$

$$W = \{x \in \partial N_i | x_{k_i+1}^2 + \dots + x_n^2 \leq (\epsilon_0)^2\}$$

Schematically, this takes the form shown in figure 10.

Note that, by construction, $q_0 \in W$. We now define a pair of vector fields, R_V and R_W , on V, W respectively, as seen in Figure 11:

$$R_V = -x_{k_i+1} \partial_{x_{k_i+1}} - \dots - x_n \partial_{x_n}$$

$$R_W = x_1 \partial_{x_1} + \dots x_{k_i} \partial_{x_{k_i}}$$

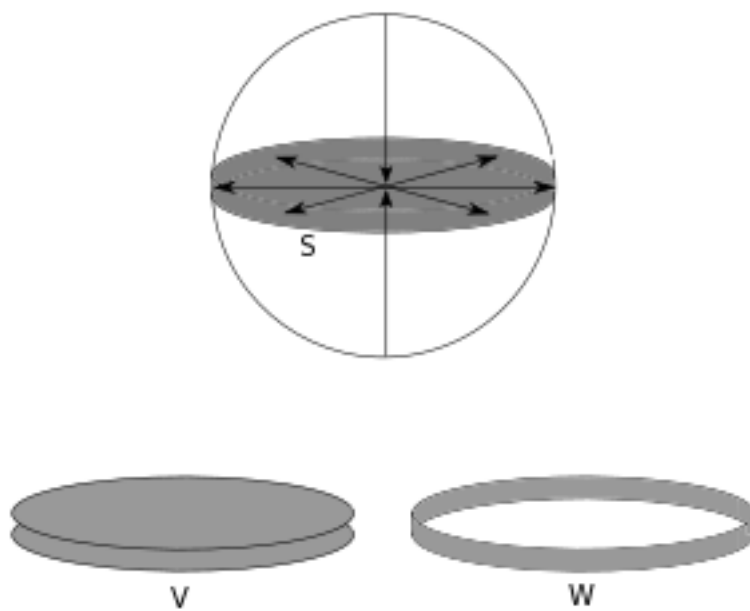


Figure 10: S, V , and W

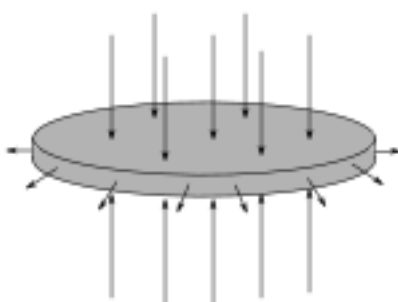


Figure 11: R_V and R_W

Observe that, using the ambient metric g of $N_i \subset \mathbb{R}^n$:

$$\nabla \psi_i^* f_1|_V \cdot R_V = -x_{k_i+1}^2 - \dots - x_n^2 < 0$$

$$\nabla \psi_i^* f_1|_W \cdot R_W = -x_1^2 - \dots - x_{k_i}^2 < 0$$

What this means is that the flow lines of $\nabla \psi_i^* f_1$ may enter S through V and leave through W , but not the other way around.

Now observe that, if we pull f_2 back to N_i with ψ_i :

$$\nabla \psi_i^* f_2|_V \cdot R_V = \nabla \psi_i^* f_1|_V \cdot R_V + (\nabla \psi_i^* f_2 - \psi_i^* \nabla f_1)|_V \cdot R_V$$

$$\nabla \psi_i^* f_2|_W \cdot R_W = \nabla \psi_i^* f_1|_W \cdot R_W + (\nabla \psi_i^* f_2 - \nabla \psi_i^* f_1)|_W \cdot R_W$$

And we know that:

$$|\nabla \psi_i^* f_2 - \nabla \psi_i^* f_1| \leq |\psi_i|_{C^1} |f_2 - f_1|_{C^1}$$

Therefore, if δ is small enough:

$$\nabla \psi_i^* f_2|_V \cdot R_V < 0 \text{ and } \nabla \psi_i^* f_2|_W \cdot R_W < 0 \quad (\text{B.3})$$

For all $f_2 \in \mathcal{F}(f_1, \delta)$. And, for δ small enough, $p'_i \in S$. What this means is that the flow lines of $\nabla \psi_i^* f_2$ for any $f_2 \in \mathcal{F}(f_1, \delta)$ will enter S through V and leave through W , but not vice-versa, if δ is small enough. This means, first, that $\mathcal{D}_{f_2}(p'_i) \cap \partial N_i \subset W$. In addition, it means that $\mathcal{A}_{f_2}(p'_i) \cap \partial N_i \not\subset W$.

Define $Z = [\mathcal{D}_{f_2}(p'_i) \cap W] \in H_{k_i-1}(W)$, where $H_*(W)$ is the singular homology of W with \mathbb{Z} coefficients. Since W is diffeomorphic to $S^{k_i-1} \times [-1, 1]^{n-k_i+1}$, we know that $H_{k_i-1}(W) \cong \mathbb{Z}$. If Z is trivial as an element of the homology of H_{k_i-1} , then it bounds some disk $Y \subset W$, so that $\partial Y = Z$. Since the descending manifold of p'_i is transverse to the ascending manifold at p'_i , and intersects it at no other point, that implies that $\mathcal{A}_{f_2}(p'_i) \cap \partial N_i \subset Y \subset W$, which is a contradiction. Therefore, Z cannot be trivial. This, in turn, implies that for any choice of $(x_1, \dots, x_k, 0, \dots, 0) \in W$, there exists some point $(x_1, \dots, x_k, x_{k+1}, \dots, x_n) \in \mathcal{D}_{f_2}(p'_i) \cap W$.

Therefore, if δ is small enough to ensure equation B.3 holds, then the intersection $\psi_i^{-1}(U_1) \cap \mathcal{D}_{f_2}(p'_i)$ will be non-empty, and contain at least some open k_i -disk. Call the image of this disk in M D_1 .

Now consider $\phi_1^{-1}(D_1)$, as shown in figure 12. From the Tubular Flow Theorem, we know that, in $[-1, 1]^n$, the pullback of ∇f_1 has the form:

$$\nabla(\phi_1^* f_1) = \lambda_1(x) \partial_{x_1}$$

Where $\lambda_1(x) > 0$. Since $[-1, 1]^n$ is compact, we can find $l_1 > 0$ such that $\lambda_1(x) > l_1$ for all x . Let q_1 be the point where $\phi_1^{-1} \circ \gamma$ leaves $[-1, 1]^n$. We also know that:

$$\nabla(\phi_1^* f_2) = \nabla(\phi_1^* f_1) + \nabla(\phi_1^*(f_2 - f_1)) \quad (\text{B.4})$$

$$|\nabla(\phi_1^*(f_2 - f_1))| \leq \delta \|\phi_1\|_{C^1} \quad (\text{B.5})$$



Figure 12: Preimage of U_1

Now, pick an arbitrary point in $\phi_1^{-1}(D_1)$, and let $\gamma^1 : [0, T_1] \rightarrow [-1, 1]^n$ be the flow line beginning at that point for $\nabla(\phi_1^* f_2)$, parameterized by $(\gamma^1)' = \nabla(\phi_1^* f_2)$, such that $\gamma^1(T_1) \in \partial[-1, 1]^n$. Then:

$$\frac{d\gamma^1}{dt} = \lambda(\gamma^1(t))\partial_{x_1} + \nabla\phi_1^*(f_2 - f_1) \quad (\text{B.6})$$

Let γ_i^1 denote the x_i coordinate of γ^1 . Then equations B.4, B.5, and B.6 together imply that:

$$\begin{aligned} \frac{d\gamma_1^1}{dt} &> l_1 - \|\phi_1\|_{C^1}\delta \\ \left| \frac{d\gamma_i^1}{dt} \right| &< \|\phi_1\|_{C^1}\delta \text{ for } i \neq 1 \end{aligned}$$

Therefore, since the width of $[-1, 1]^n$ is 2, we can bound T_1 by:

$$T_1 < \frac{2}{l_1 - \|\phi_1\|_{C^1}\delta}$$

And, for $i \neq 1$:

$$|\gamma_i^1(T_1) - \gamma_i^1(0)| < \|\phi_1\|_{C^1}\delta T_1 < \frac{2\|\phi_1\|_{C^1}\delta}{l_1 - \|\phi_1\|_{C^1}\delta}$$

Therefore, for any choice of $\epsilon_1 > 0$, if ϵ_0 and δ are small enough, then $\gamma^1(T_1)$ will be within ϵ_1 of q_1 . Since this is an open condition, if ϵ_0 and δ are small enough, there will be some k_i -disk in $\phi_1^{-1}(\mathcal{D}_{f_2}(p'_i))$ within ϵ_1 of q_1 . Since $q_1 \in \phi_1^{-1}(U_2)$, this means that, if ϵ_1 is small enough, this k_i -disk will lie within $\phi_1^{-1}(U_2)$. Call the image of this disk in M D_2 .

Now repeat this process for D_2 in $\phi_2^{-1}(U_2)$. If ϵ_1 and δ are small enough, there will be some D_3 within ϵ_2 of q_2 , which is the point where γ leaves U_2 .

Then repeat the process for D_3 in $\phi_3^{-1}(U_3)$, and so on, until we reach an open k_i -disk $D_m \subset \psi_j(N_j)$. Then $\psi_j^{-1}(D_m)$ will be a k_i -disk in N_j . Analogous to N_i , we can define:

$$\begin{aligned} S' &= \left\{ x \in N_j \mid x_1^2 + \dots + x_{k_j}^2 < \epsilon_{m+1}^2 \right\} \\ V' &= \left\{ x \in N_j \mid x_1^2 + \dots + x_{k_j}^2 = \epsilon_{m+1}^2 \right\} \\ W' &= \left\{ x \in \partial N_j \mid x_1^2 + \dots + x_{k_j}^2 \leq \epsilon_{m+1}^2 \right\} \end{aligned}$$

And, if δ is small enough, then $p'_j \in S'$, the ascending manifold of p'_j intersects ∂N_j in W' , and the descending manifold of p'_j intersects $\partial S'$ in V' . Since the pullback of γ lies within $\mathcal{A}_{f_1}(p_j)$, which is within S' , then if ϵ_m is small enough, $\psi_j^{-1}(D_m)$ will lie within S' . Generically, the intersection of $\psi_j^{-1}(D_m)$ with $\mathcal{A}_{f_2}(p'_j)$ will be $(k_i - k_j)$ -dimensional.

Therefore, for any choice of flow line γ from p_i to p_j for f_1 , we can find some δ_γ such that if $\delta < \delta_\gamma$, then for any $f_2 \in \mathcal{F}(f_1, \delta)$ there exists some flow line from p'_i to p'_j for f_2 that lies within ϵ of γ . Since the set of flow lines emerging from p_i is diffeomorphic to S^{k_i-1} , we can find some δ_i such that if $\delta < \delta_i$, then for any flow line γ emerging from p_i and for any $f_2 \in \mathcal{F}(f_1, \delta)$ there exists some flow line from p'_i to p'_j for f_2 that lies within ϵ of γ . And since there are only finitely many critical points of f_1 , we can find δ such that, if $f_2 \in \mathcal{F}(f_1, \delta)$, this holds for any critical point. This concludes the proof.

Lemma 3.2: *Let $f : M \rightarrow \mathbb{R}$ be a Morse function, where M is a closed manifold. Let Q be a compact codimension-0 subset of M that includes no critical points of f . Then for any $\epsilon > 0$ there exists $\delta > 0$ such that for any critical point q and any points $p_1, p_2 \in Q$, if:*

$$d(p_1, p_2) < \delta, \text{ and}$$

$$p_1, p_2 \text{ lie in the same component of } \mathcal{A}_f(q)$$

Then:

$$d(\mathcal{D}_f(p_1), \mathcal{D}_f(p_2)) < \epsilon$$

And, for any $\epsilon > 0$, there exists $\delta > 0$ such that, for any points $p_1, p_2 \in Q$, if

$$d(p_1, p_2) < \delta, \text{ and}$$

$$p_1, p_2 \text{ lie in the same component of } \mathcal{D}_f(q)$$

Then:

$$d(\mathcal{A}_f(p_1), \mathcal{A}_f(p_2)) < \epsilon$$

Proof: We begin by proving that, given a *specific* point $p_1 \in Q$, we can find δ depending on p_1 such that if $d(p_1, p_2) < \delta$, then $d(\mathcal{D}_f(p_1), \mathcal{D}_f(p_2)) < \epsilon$.



Figure 13: γ and its covering by tubular flow neighborhoods and $\psi_0(N_0)$

We will then use compactness to extend the result to all of Q . We prove the statement for the descending manifold, since $\mathcal{A}_f(p) = \mathcal{D}_{-f}(p)$.

Assume p_1 is in the ascending manifold of q . Let $\psi_0 : N_0 \rightarrow M$ be a Morse neighborhood of q , so that N_0 is an open n -ball and the image $\psi_0(N_0)$ is contained in an open ball of radius $\epsilon_0 \leq \frac{1}{2}\epsilon$ centered at q . Let x_1, \dots, x_n be the coordinates of $N_0 \subset \mathbb{R}^n$. By definition, $\psi_0^*f = f(q_1) \pm x_1^2 \pm \dots \pm x_n^2$. Therefore:

$$\nabla(\psi_0^*f) = \pm 2x_1\partial_{x_1} \pm \dots \pm 2x_n\partial_{x_n}$$

This means that if $p \in N_0$ is in the ascending manifold of $q = 0$, its descending manifold in N_0 is simply a straight line starting at p and ending at 0.

Therefore, for every point in the image of $\psi_0(N_0)$ that lies in the ascending manifold of q , its descending manifold is contained in $\psi_0(N_0)$.

Now, let $\gamma : [0, l] \rightarrow M$ be a parameterization of the descending manifold of p_1 by arc length, $\gamma(0) = p_1, \gamma(l) = q$. By the Tubular Flow Theorem, for every point $\gamma(t), 0 \leq t < l$, we can find a local neighborhood $\tilde{\psi}_t : N'_t \rightarrow M$ of $\gamma(t)$, where N'_t is diffeomorphic to $(-1, 1)^n$, where the flow lines of ∇f are mapped to lines with constant x_2, \dots, x_n coordinates, and where the image $\tilde{\psi}_t(N'_t)$ is contained in a $\frac{1}{2}\epsilon$ -neighborhood of the image of γ . Then, with the addition of $\psi_0(N_0)$, $\tilde{\psi}_t(N'_t)$ form an infinite cover of the image of γ , as shown in figure 13. Since the image of γ is compact, we can find a finite subcover $N_0, N'_{t_1}, \dots, N'_{t_k}, t_1 > t_2 > \dots > t_k$ - that is, $\psi_1(N_1)$ is close to the critical point q_1 , while $\psi_k(N_k)$ contains our starting point p_1 . We write $N_i = N'_{t_i}$ and let ψ_i denote the embedding $\tilde{\psi}_{t_i} : N'_{t_i} \rightarrow M$ for succinctness.

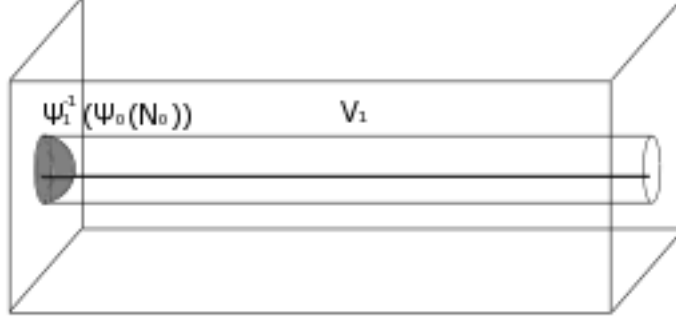


Figure 14: N_1, W_1, V_1 , and γ

Now, consider the preimage $\psi_1^{-1}(\psi_0(N_0))$. As shown in Figure 14, let W_1 denote the projection of $\psi_1^{-1}(\psi_0(N_0))$ to $(-1, 1)^{n-1}$, dropping the x_1 coordinate, and let V_1 denote the ascending manifold of $\psi_1^{-1}(\psi_0(N_0))$ in N_1 . Since the pull-back of the vector field ∇f to N_1 is $\lambda(x)\partial_{x_1}, \lambda(x) > 0$, V_1 will be diffeomorphic to $(-1, 1) \times W_1$:

$$V_1 = \{(x_1, x_2, \dots, x_n) : (x_2, \dots, x_n) \in W_1\}$$

Note that V_1 will contain $\psi_1^{-1}(\mathcal{D}_f(p_1))$. Note also that, since ψ_0 is a diffeomorphism onto its image and its domain is open, $\psi_0(N_0)$ is open. Therefore $\psi^{-1}(\psi_0(N_0))$ is open, so $W_1 \subset (-1, 1)^{n-1}$ is open, so V_1 is open.

Now consider $\psi_2^{-1}(\psi_1(V_1))$. Let W_2 be the projection of $\psi_2^{-1}(\psi_1(V_1))$ that drops the x_1 coordinate, let V_2 be the ascending manifold of $\psi_2^{-1}(\psi_1(V_1))$ in N_2 , and observe that $V_2 = (-1, 1) \times W_2$, that V_2 contains $\psi_2^{-1}(\mathcal{D}_f(p_1))$, and that V_2 is open. Continue repeating this process until you reach V_k .

Now consider $\psi_k(V_k)$. Since ψ_k is a diffeomorphism onto its image, $\psi_k(V_k)$ will be open. Since V_k contains the $\psi_k^{-1}(\mathcal{D}_f(p_1))$, $\psi_k(V_k)$ contains p_1 . Therefore, $\psi_k(V_k)$ contains an open ball $B_\delta(p_1)$ of radius δ around p_1 .

Suppose $p_2 \in B_\delta(p_1)$, and p_2 lies in the same component of $\mathcal{D}_f(q)$ as p_1 . Then $\psi_k^{-1}(p_2) \in V_k$, so its descending manifold in N_k lies in V_k . We can continue forward from here, showing that the preimage in N_i of the descending manifold $\mathcal{D}_f(p_2)$ lies in V_i for all i . Then, once we reach N_0 , since $p_2 \in \mathcal{D}_f(q)$, the descending manifold terminates in a curve to q . Since N_i lie in an ϵ -neighborhood of $\mathcal{D}_f(p_1)$, we may conclude that, for any $p \in Q$, we can find $\delta(p)$ such that, if:

$$d(p, p_2) < \delta(p), \text{ and}$$

$$p, p_2 \text{ are in the same component of } \mathcal{D}_f(q)$$

Then:

$$d(\mathcal{D}_f(p), \mathcal{D}_f(p_2)) < \frac{1}{2}\epsilon$$

Let $B_{\frac{1}{2}\delta(p)}(p)$ denote an open ball of radius $\frac{1}{2}\delta(p)$ around p , where $p \in Q$. These balls form an infinite open cover of Q . Since Q is compact, we can find a finite subcover $B_{\frac{1}{2}\delta(p'_1)}(p'_1), \dots, B_{\frac{1}{2}\delta(p'_m)}(p'_m)$. Define:

$$\delta = \frac{1}{2} \min_i \delta(p'_i)$$

Suppose $p_1, p_2 \in Q$ and $d(p_1, p_2) < \delta$. p_1 must be contained in some ball $B_{\frac{1}{2}\delta(p'_k)}(p'_k)$. Then, the distance from p_2 to p'_k is:

$$d(p_2, p'_k) \leq d(p_2, p_1) + d(p_1, p'_k) \leq \delta + \frac{1}{2}\delta(p'_k) \leq \delta(p'_k)$$

So $p_2 \in B_{\frac{1}{2}\delta(p'_k)}(p'_k)$. Therefore:

$$d(\mathcal{D}_f(p_1), \mathcal{D}_f(p'_k)) < \frac{1}{2}\epsilon$$

$$d(\mathcal{D}_f(p_2), \mathcal{D}_f(p'_k)) < \frac{1}{2}\epsilon$$

Which in turn implies that:

$$d(\mathcal{D}_f(p_1), \mathcal{D}_f(p_2)) < \epsilon$$

This concludes the proof.

Lemma 3.3: *Let $f_1 : M \rightarrow \mathbb{R}$ be a Morse function, where M is a compact manifold. For any choice of $\epsilon > 0$ and any compact codimension-0 submanifold $Q \subset M$ that lies in the descending manifold of maxima of f_1 and contains no critical points of f_1 , there exists δ such that, for any generic choice of $f_2 \in \mathcal{F}(f_1, \delta)$ and any $p \in Q$, the ascending manifold of p for ∇f_2 will lie within an ϵ -neighborhood of the ascending manifold of p for ∇f_1 .*

Proof: The proof of this lemma is essentially analogous to the proof of lemma 3.1. Pick any point $p \in Q$, and let $\gamma : [0, l] \rightarrow M$ be the flow line from p to its maximum q for $-\nabla f_1$ parameterized by arc length, so that $\gamma(0) = p, \gamma(l) = q$. Let $\Psi : U(q) \rightarrow M$ be a Morse neighborhood of q of radius less than ϵ . Find a cover of $\text{Im } \gamma - \Psi(U(q))$ by Tubular Flow neighborhoods, $\phi_1 : [-1, 1]^n \rightarrow U_1, \dots, \phi_m : [-1, 1]^n \rightarrow U_m$. Restrict δ to be small enough that the critical points of any $f_2 \in \mathcal{F}(f_1, \delta)$ will be in one-to-one correspondence with the critical points of f_1 , and the critical point of f_2 corresponding to q will lie inside $\Psi(U(q))$. Note that we require f_2 to always be generic, to ensure that p will still lie in the descending manifold of the maximum of f_2 corresponding to q .

Consider $-\nabla(\phi_1^* f_1)$. By the Tubular Flow Theorem, this will equal:

$$\nabla(-\phi_1^* f_1) = \lambda_1(x) \partial_{x_1}$$

For some function $\lambda_1 : [-1, 1]^n \rightarrow \mathbb{R}, \lambda_1(x) > 0$. Since $[-1, 1]^n$ is compact, we can find $l_1 > 0$ such that $\lambda_1(x) > l_1$ for all x . Let q_1 be the point where $\phi_1^{-1} \circ \gamma$ leaves $[-1, 1]^n$. We know that, for $f_2 \in \mathcal{F}(f_1, \delta)$:

$$\nabla(\phi_1^* f_2) = \nabla(\phi_1^* f_1) + \nabla(\phi_1^*(f_2 - f_1)) \quad (\text{B.7})$$

$$|\nabla(\phi_1^*(f_2 - f_1))| \leq \delta \|\phi_1\|_{C^1} \quad (\text{B.8})$$

Define $\gamma_{f_2} : [0, T_1] \rightarrow [-1, 1]^n$ to be the flow line beginning at $\phi_1^{-1}(p)$ and ending on $\partial[-1, 1]^n$ for $-\nabla(\phi_1^* f_2)$, parameterized by $(\gamma_{f_2})' = -\nabla(\phi_1^* f_2)$. Then:

$$-\frac{d\gamma_{f_2}}{dt} = \lambda(\gamma_{f_2}(t))\partial_{x_1} + \nabla(\phi_1^*(f_2 - f_1)) \quad (\text{B.9})$$

Let $(\gamma_{f_2})_i$ denote the x_i coordinate of γ_{f_2} . Then equations B.7, B.8, and B.9 together imply that:

$$\begin{aligned} -\frac{d(\gamma_{f_2})_1}{dt} &> l_1 - \|\phi_1\|_{C^1}\delta \\ \left| -\frac{d(\gamma_{f_2})_i}{dt} \right| &< \|\phi_1\|_{C^1}\delta \text{ for } i \neq 1 \end{aligned}$$

Therefore, since the width of $[-1, 1]^n$ is 2, for any $f_2 \in \mathcal{F}(f_1, \delta)$ we can bound T_1 by:

$$T_1 < \frac{2}{l_1 - \|\phi_1\|_{C^1}\delta}$$

And, for $i \neq 1$:

$$|(\gamma_{f_2}(T_1))_i - (\gamma_{f_2}(0))_i| < \|\phi_1\|_{C^1}\delta T_1 < \frac{2\|\phi_1\|_{C^1}\delta}{l_1 - \|\phi_1\|_{C^1}\delta}$$

Since the pullback of $\gamma(t)$ to $[-1, 1]^n$ has constant x_2, \dots, x_n coordinates for all t , we may conclude that, for any choice of $\epsilon_1 > 0$, if δ is small enough then $\gamma_{f_2}(T_1)$ will lie within ϵ_1 of q_1 for any $f_2 \in \mathcal{F}(f_1, \delta)$ - and, indeed, all points of γ_{f_2} will lie within ϵ_1 of γ within U_1 . And, if ϵ_1 is small enough, $\phi_1(\gamma_{f_2}(T_1))$ will lie inside $\phi_2([-1, 1]^n)$ for any choice of $f_2 \in \mathcal{F}(f_1, \delta)$.

Now define D_1 to be a closed ϵ_1 -ball around $\phi_1(q_1)$ in M , and consider $\phi_2^{-1}(D_1) \subset [-1, 1]^n$. Define q_2 to be the point in $[-1, 1]^n$ where $\phi_2^{-1} \circ \gamma$ leaves the cube. For any point $p_2 \in \phi_2^{-1}(D_1)$, define $\gamma_{f_2}^{p_2} : [0, T_2] \rightarrow [-1, 1]^n$ to be the flow line of $-\nabla\phi_2^* f_2$ starting at p_2 and ending on $\partial[-1, 1]^n$. By an analogous argument we can again bound:

$$|(\gamma_{f_2}^{p_2})_i(T_2) - (\gamma_{f_2}^{p_2})_i(0)| < \frac{2\|\phi_2\|_{C^1}\delta}{l_2 - \|\phi_2\|_{C^1}\delta}$$

Where:

$$\begin{aligned} l_2 &< \lambda_2(x) \text{ for all } x \in [-1, 1]^n \\ -\nabla(\phi_2^* f_1) &= \lambda_2(x)\partial_{x_1} \end{aligned}$$

Since D_1 is compact, for any choice of $\epsilon_2 > 0$, for any point $p_2 \in \phi_2^{-1}(D_1)$, and for any $f_2 \in \mathcal{F}(f_1, \delta)$, if δ is small enough then the flow line of $-\nabla(\phi_2^* f_2)$ starting at p_2 and ending on $\partial[-1, 1]^n$ will be within $\epsilon_1 + \epsilon_2$ of q . Therefore, for any $\epsilon_1, \epsilon_2 > 0$ and any $f_2 \in \mathcal{F}(f_1, \delta)$, if δ is small enough then the flow line of $-\nabla f_2$ beginning at p and ending on $\phi_2(\partial[-1, 1]^n)$ will be within $\epsilon_1 + \epsilon_2$ of q .

Proceeding in this way, we can ultimately show that, if δ is small enough, then for any $f_2 \in \mathcal{F}(f_1, \delta)$, the flow line of $-\nabla f_2$ beginning at p and ending on $\phi_m(\partial[-1, 1]^n)$ will lie within $\epsilon_1 + \dots + \epsilon_m$ of q . If we choose $\epsilon_1 + \dots + \epsilon_m$ to be small enough, then the flow line will end on $\partial(\Psi(U(q)))$, where $\Psi : U(q) \rightarrow M$ is the Morse neighborhood of the maximum q .

Now, consider $U(q)$. The pullback of f_1 to $U(q)$ is:

$$\Psi^* f_1 = f_1(q) - x_1^2 - \dots - x_n^2$$

Therefore, if we define $R = x_1 \partial_{x_1} + \dots + \partial_{x_n}$, then in the ambient metric of $U(q)$:

$$(R \cdot \nabla(\Psi^* f_1))|_{\partial U(q)} < 0$$

And, if we choose δ small enough, we can therefore ensure that:

$$(R \cdot \nabla(\Psi^* f_2))|_{\partial U(q)} < 0$$

Therefore, if δ is small enough, the ascending manifold of any point on $\partial U(q)$ for any $\nabla f_2, f_2 \in \mathcal{F}(f_1, \delta)$, will lie inside $U(q)$. And since $U(q)$ has radius less than ϵ , the image of this ascending manifold in M must lie within ϵ of q .

Therefore, for any $p \in Q$ we can find $\delta(p)$ such that, for any generic $f_2 \in \mathcal{F}(f_1, \delta(p))$, the ascending manifold of p for f_2 lies within ϵ of the ascending manifold of p for f_1 . And, since Q is compact, we can therefore find δ such that $\delta(p) < \delta$ for all p . This concludes the proof.

Lemma 3.4: *Let $\Lambda \subset J^1(M)$ be a front-generic Legendrian submanifold whose front projection is defined by sheet functions $f_1 : U_1 \rightarrow \mathbb{R}, \dots, f_m : U_m \rightarrow \mathbb{R}$, where $U_1, \dots, U_m \subset M$. Let $\hat{\Lambda} \subset J^1(M)$ be a second front-generic Legendrian submanifold whose front projection is defined by sheet functions $\hat{f}_1 : U_1 \rightarrow \mathbb{R}, \dots, \hat{f}_m : U_m \rightarrow \mathbb{R}$. Then, for any choice of $\epsilon > 0$, there exists $\delta > 0$ such that, if:*

$$\left\| f_i - \hat{f}_i \right\|_{C^1} < \delta \text{ for all } i$$

Then, there exists a bijection between the rigid gradient flow trees Λ with one positive Reeb chord and the rigid gradient flow trees of $\hat{\Lambda}$ with one positive Reeb chord, such that a tree Γ of Λ shares the same Reeb chords with its corresponding tree $\hat{\Gamma}$ of $\hat{\Lambda}$, and such that the projection of $\hat{\Gamma}$ to M lies within an ϵ -neighborhood of the projection of Γ to M .

Proof: Restrict δ to be small enough that the Reeb chords of $\hat{\Lambda}$ are in one-to-one correspondence with the Reeb chords Λ . Consider some rigid gradient flow tree Γ of Λ , which has only one positive Reeb chord and whose vertices are

only Reeb chords and Y^0 vertices. Begin by considering the case where Γ has no 2-valent vertices at Reeb chords.

Γ is equipped with a natural direction, given by the direction of the gradient flow functions. Label each vertex of Γ with the number of edges of the longest directed path from that vertex to a negative Reeb chord - so, e.g., Reeb chords would be labeled with 0. Let V_i denote the set of vertices of Γ which are labeled with i . For a Y^0 vertex $v \in V_i$, define w_v^1, w_v^2 to be the pair of vertices in Γ such that there is an edge in Γ connecting v to them, directed from v to w_v^1, w_v^2 . Define $f_{w_v^j}, g_{w_v^j}$ to be the sheet height functions such that the edge from v to w_v^j flows on the gradient flow $-\nabla(f_{w_v^j} - g_{w_v^j})$. Let $\hat{f}_{w_v^j}, \hat{g}_{w_v^j}$ denote the corresponding sheet height functions of $\hat{\Lambda}$.

We will define sets $Y(v), Z(v)$ iteratively. We begin by defining, for any $v \in V_0$, $Y(v) = \{v\}$. Then, for any v - not just in V_0 - we define:

$$Z(v) = \mathcal{A}_{f_v - g_v}(Y(v))$$

And for $v \notin V_0$, we define:

$$Y(v) = Z(w_v^1) \cap Z(w_v^2)$$

Now, for any Reeb chord of Λ designated by $v \in V_0$, there is a corresponding Reeb chord of $\hat{\Lambda}$. We denote this Reeb chord by \hat{v} , and we define:

$$\hat{Y}(v) = \{\hat{v}\}$$

Then, for $v \notin V_0$, we define:

$$\hat{Z}(\mathcal{A}_{\hat{f}_v - \hat{g}_v}(\hat{Y}(v)))$$

$$\hat{Y}(v) = \hat{Z}(w_v^1) \cap \hat{Z}(w_v^2)$$

Now, consider any $v \in V_0$. By Lemma 3.1, if $\delta > 0$ is small enough, then $\hat{Z}(v)$ lies within an ϵ_0 -neighborhood of $Z(v)$ for all $v \in V_0$, for any choice of $\epsilon_0 < \epsilon$.

Then, let $N_{\epsilon_0}(Z(v))$ denote an ϵ_0 -neighborhood of $Z(v)$. For $v \in V_1$, consider $N_{\epsilon_0}(Z(w_v^1)) \cap N_{\epsilon_0}(Z(w_v^2))$. If ϵ_0 is small enough, then for any choice of $\delta_1 > 0$, every point in $(N_{\epsilon_0}(Z(w_v^1)) \cap N_{\epsilon_0}(Z(w_v^2)))$ will lie within a δ_1 -neighborhood of $Z(w_v^1) \cap Z(w_v^2) = Y(v)$. Since $\hat{Z}(w_v^1) \subset N_{\epsilon_0}(Z(w_v^1)), \hat{Z}(w_v^2) \subset N_{\epsilon_0}(Z(w_v^2))$, we know $\hat{Y}(v)$ is non-empty, and $\hat{Y}(v) \subset N_{\epsilon_0}(Z(w_v^1)) \cap N_{\epsilon_0}(Z(w_v^2))$. From this we conclude that if ϵ_0 is small enough, then $\hat{Y}(v)$ lies within a δ_1 -neighborhood of $Y(v)$.

Now consider $\hat{Z}(v)$ for $v \in V_1$. We know that $Y(v)$ must be at least codimension-1, because if they are codimension-0, then Γ is not rigid; and since δ is small enough that our Reeb chords correspond, this means $\hat{Y}(v)$ must also be at least codimension-1. Therefore, we can pick $Q \subset M$ containing $Y(v)$ but not containing any Reeb chords of $f_{v_i} - g_{v_i}$. Therefore, by Lemma 3.3, if δ_1 is small enough, then ascending manifold of $\hat{Y}(v)$ for $-\nabla(f_v - g_v)$ will lie within $\frac{1}{2}\epsilon_1$ of $\hat{Z}(v)$. And by Lemma 3.2, if δ_1 is small enough, the ascending manifold

of $\widehat{Y}(v)$ for $-\nabla(f_v - g_v)$ will lie within $\frac{1}{2}\epsilon_1$ of $Z(v)$. Therefore, if δ_1 is small enough, then $\widehat{Z}(v)$ will lie within ϵ_1 of $Z(v)$.

Now, let $v \in V_2$, and consider $N_{\epsilon_1}(Z(w_v^1)) \cap N_{\epsilon_1}(Z(w_v^2))$. Note that $w_v^1, w_v^2 \in V_1 \cup V_0$. If ϵ_1 is small enough, then for any choice of $\delta_2 > 0$, every point in $(N_{\epsilon_0}(Z(w_v^1)) \cap N_{\epsilon_0}(Z(w_v^2)))$ will lie within δ_2 of $Z(w_v^1) \cap Z(w_v^2) = Y(v)$. Since $\widehat{Y}(v) \subset N_{\epsilon_1}(Z(w_v^1)) \cap N_{\epsilon_1}(Z(w_v^2))$, we conclude that if ϵ_1 is small enough, then $\widehat{Y}(v)$ is non-empty and lies within a δ_2 -neighborhood of $Y(v_i)$.

We keep repeating this process until we reach $Z(v_a), \widehat{Z}(v_a)$, where a is the positive Reeb chord of Γ , and v_a is the Y^0 vertex connected by an edge to a . By this process, we show that $\widehat{Z}(v_a)$ lies within an ϵ_m -neighborhood of $Z(v_a)$. Let \hat{a} denote the Reeb chord of $\hat{\Lambda}$ corresponding to a .

Now define $X(a), \widehat{X}(a)$ to be:

$$X(a) = \mathcal{D}_{-(f_{v_a} - g_{v_a})}(a) \cap Z(v_a)$$

$$\widehat{X}(a) = \mathcal{D}_{-(\hat{f}_{v_a} - \hat{g}_{v_a})}(\hat{a}) \cap \widehat{Z}(v_a)$$

Since Γ is rigid, $X(a), \widehat{X}(a)$ must be one-dimensional, and $X(a)$ must be equal to the image in the base space of the edge of Γ emerging from a . By Lemma 3.1, if δ is small enough, then $\mathcal{D}_{-(f_{v_a} - g_{v_a})}(a)$ will lie within ϵ_{m+1} of $\mathcal{D}_{-(\hat{f}_{v_a} - \hat{g}_{v_a})}(\hat{a})$. And we already know that if $\delta, \epsilon_1, \dots, \epsilon_m$ are small enough, then $Z(v_a)$ will lie within ϵ_{m+1} of $\widehat{Z}(v_a)$. Therefore, if ϵ_{m+1} is small enough, then $X(a)$ will lie within δ_m of $\widehat{X}(a)$. And, since $X(a), \widehat{X}(a)$ are one-dimensional, we know that $\partial X(a) = \{a, v_a\}$ and $\partial \widehat{X}(a) = \{\hat{a}, \hat{v}_a\}$.

Now define $X_i(v_a), \widehat{X}_i(v_a)$ to be:

$$X_i(v_a) = \mathcal{D}_{-(f_{w_{v_a}^i} - g_{w_{v_a}^i})}(v_a) \cap Z(w_{v_a}^i)$$

$$\widehat{X}_i(v_a) = \mathcal{D}_{-(\hat{f}_{w_{v_a}^i} - \hat{g}_{w_{v_a}^i})}(\hat{v}_a) \cap \widehat{Z}(w_{v_a}^i)$$

Once again, since Γ is rigid, we can conclude that $X_i(v_a), \widehat{X}_i(v_a)$ are one-dimensional. We know that $Z(w_{v_a}^i)$ lies within ϵ_{m-1} of $\widehat{Z}(w_{v_a}^i)$. And, by Lemma 3.2, if δ, δ_m are small enough, $\mathcal{D}_{-(f_{w_{v_a}^i} - g_{w_{v_a}^i})}(v_a)$ will lie within ϵ_{m+2} of $\mathcal{D}_{-(\hat{f}_{w_{v_a}^i} - \hat{g}_{w_{v_a}^i})}(\hat{v}_a)$. Therefore, if $\delta, \epsilon_1, \dots, \epsilon_m, \epsilon_{m+1}$ are small enough, $\widehat{X}_i(v_a)$ will lie within δ_{m+1} of $X_i(v_a)$. Repeat this process until we reach the Reeb chords. The trace of the $\widehat{X}_i(v)$ form a unique rigid gradient flow tree $\hat{\Gamma}$ of $\hat{\Lambda}$ with Reeb chords corresponding to the Reeb chords of Γ , and lying within an ϵ -neighborhood of Γ .

Now consider the case where Γ has a two-valent Reeb chord. We can break Γ into sub-trees $\Gamma_1, \dots, \Gamma_k$ at every two-valent Reeb chord. We can then repeat the process.

Therefore, for every rigid Γ there exists some δ_Γ such that if $\|f_i - \hat{f}_i\|_{C^1} < \delta$ for all i , then there exists a unique rigid gradient flow tree $\hat{\Gamma}$ of $\hat{\Lambda}$ that lies within an ϵ -neighborhood of Γ . Since there are only finitely-many rigid flow trees, we

can therefore find δ such that, if $\|f_i - \hat{f}_i\|_{C^1} < \delta$ for all i , then $\hat{\Gamma}$ lies within an ϵ -neighborhood of Γ for all Γ . This completes the proof.

References

- [1] Audin, Michele, and Lafontaine, Jacques (Eds.). (1994). *Holomorphic Curves in Symplectic Geometry. Progress in Mathematics*, Vol. 117. Berlin, Germany: Birkhauser Basel.
- [2] Chekanov, Yuri. (2002). Differential Algebra of Legendrian Links. *Inventiones Mathematicae*, Vol. 150, 441-483.
- [3] Driver, Bruce K. (2003). *Analysis Tools with Applications*. Berlin, Germany: Springer.
- [4] Ekholm, T., Etnyre, J., and Sullivan, M. (2005). The Contact Homology of Legendrian Submanifolds in \mathbb{R}^{2n+1} . *Journal of Differential Geometry*, Vol. 71, 177-305.
- [5] Ekholm, T., Etnyre, J., and Sullivan M. (2005). Non-Isotopic Legendrian Submanifolds in \mathbb{R}^{2n+1} . *Journal of Differential Geometry*, Vol. 71, 85-128.
- [6] Ekholm, T. (2007). Morse Flow Trees and Legendrian Contact Homology in 1-Jet Spaces. *Geometry & Topology* Vol. 11, 1083-1224.
- [7] Ekholm, T., Honda, K., and Kalman, T. (2012). Legendrian Knots and Exact Lagrangian Cobordisms. arXiv:1212.1519v3 [math.SG].
- [8] Fuchs, Dimitry. (2003). Chekanov-Eliashberg Invariant of Legendrian Knots: Existence of Augmentations. *Journal of Geometry and Physics*, Vol. 47, 43-65.
- [9] Harper, John G., and Sullivan, Michael G. (2014). A Bordered Legendrian Contact Algebra. *Journal of Symplectic Geometry*, Vol. 12 no. 2, 237-255.
- [10] MacDuff, Dusa, and Salamon, Dietmar. (1995). *Introduction to Symplectic Topology*. Oxford, UK: Clarendon Press.
- [11] Palis, Jr., Jacob, and de Melo, Welington. (1982). *Geometric Theory of Dynamical Systems: An Introduction*. New York, NY: Springer-Verlag.
- [12] Rizell, Georgios. (2012). Legendrian Ambient Surgery and Legendrian Contact Homology. arXiv:1205.5544v5 [math.SG].
- [13] Sabloff, Joshua. (2005). Augmentations and Rulings of Legendrian Knots. *International Mathematical Research Notices*, No. 19, 1157-1180.
- [14] Sivek, Stephen. (2011). A Bordered Chekanov-Eliashberg Algebra. *Journal of Topology*, Vol. 4, 73-104.

- [15] Sternberg, Shlomo. (1964). *Lectures on Differential Geometry*. Upper Saddle River, NJ: Prentice-Hall.